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Abstract
We introduce a new fractional oscillator process which can be obtained as a solution of a stochastic differential equation with two fractional orders. Basic properties such as fractal dimension and short-range dependence of the process are studied by considering the asymptotic properties of its covariance function. By considering the fractional oscillator process as the velocity of a diffusion process, we derive the corresponding diffusion constant, fluctuation–dissipation relation and mean-square displacement. The fractional oscillator process can also be regarded as a one-dimensional fractional Euclidean Klein–Gordon field, which can be obtained by applying the Parisi–Wu stochastic quantization method to a nonlocal Euclidean action. The Casimir energy associated with the fractional field at positive temperature is calculated by using the zeta function regularization technique.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

Fractional calculus has found applications in diverse fields ranging from physical sciences, engineering to biological sciences and economics. Many of the recent advances in fractional calculus are motivated by the modern applications of fractional integro-differential equations in various fields, in particular physics. One of the main reasons for its popularity in modeling various transport properties in complex heterogeneous and disordered media is that it provides a natural setting for describing processes with memory and is fractal or multifractal in nature. For example, various versions of fractional diffusion equations and fractional Langevin-type equations have been proposed to model anomalous diffusion [1–7], and both deterministic and

stochastic fractional equations are used to describe non-Debye dielectric relaxation phenomena [6–9].

One way to obtain concrete realization of a particular fractional model is to associate it with a fractional generalization of an ordinary stochastic process. The most well-known among these fractional stochastic processes include fractional Brownian motion [10, 11] and fractional Levy motion [11, 12]. Other fractional stochastic processes include fractional Brownian motion of Riemann–Liouville type [13], fractional Ornstein–Uhlenbeck process or fractional oscillator process [14–17], etc. Works on extending stochastic processes characterized by a single index to corresponding processes with two indices or processes with variable index have recently attracted considerable interest. For example, fractional Brownian motion parametrized by a Hurst index H has recently been generalized to the bifractional Brownian motion [18, 19] characterized by two indices, with fractional Brownian motion as a special case. Other examples of stochastic processes with two indices are the class of Riesz–Bessel motions [20] and Gaussian processes with generalized Cauchy covariance (generalized Cauchy class) [21, 22]. In general, processes parametrized by two indices can provide more flexibility in their applications in physical phenomena. In particular, Riesz–Bessel motion and the process with generalized Cauchy class covariance both have the advantage that the two indices provide separate characterization of the fractal dimension or self-similar property, a local property, and the long-range dependence, a global property. This is in contrast to models based on fractional Brownian motion or fractional Brownian noise which use a single index to characterize these two properties. Models based on a stochastic process with single index seem inadequate. For example, it has been noted that fractional Gaussian noise is not suitable to model network traffic for all scales since at very small timescales the fluctuations in the traffic are no longer statistically self-similar [23].

On the other hand, there are also many natural phenomena that exhibit short memory, ranging from coding regions of DNA sequences to fluctuations of an electropore of nano size [24–29]. As a result, in order to obtain separate characterization of the local sample path regularity and the short memory property, it is desirable to extend short-range dependent fractional processes such as a fractional Ornstein–Uhlenbeck process (also known as a fractional oscillator process which is preferred as the fractional Ornstein–Uhlenbeck process has often been used for the process obtained from the Langevin equation driven by fractional Brownian motion) to the corresponding fractional process with two indices. In [30], we have attempted such an extension by using Weyl and Riemann–Liouville shifted fractional derivatives. The resulted process can be regarded as a generalization of the fractional Ornstein–Uhlenbeck process with single index [16, 17]. However, such a process does not inherit the simple form of the spectral density of its single-index counterpart. This leads to some problems in our application to modeling wind speed based on Von Kárman-type spectral density [27, 30]. One of our main aims is to obtain a fractional oscillator process with two indices so that it can be regarded as a one-dimension fractional Euclidean scalar massive field with two indices just like the case with single index. The result in [30] does not allow us to do this. The aim of this paper is to address this problem by introducing a new type of fractional Gaussian process indexed by two indices so that it can be identified as a fractional Euclidean field in one dimension, in addition to that its short-range dependence property and fractal dimension can be separately characterized. Keeping this in mind, we introduce a new fractional Langevin equation by replacing the Weyl or Riemann–Liouville fractional derivatives in [30] by the Riesz fractional derivative [31, 32]. If we consider the fractional oscillator process as a one-dimensional fractional Euclidean field, our use of Riesz derivative does not pose any problem. However, if it is to be considered as a velocity process in the sense of Ornstein–Uhlenbeck process with two fractional indices, one may face a difficulty

in giving a causal interpretation. Nevertheless, this would not deny the possibility of using this fractional oscillator process for modeling short memory processes [24–29]. The ability to identify the fractional oscillator process with a fractional Euclidean massive field permits us to perceive the statistical interpretation of the fractional Klein–Gordon field, and hence the calculation of quantity such as its partition energy and the associated Casimir energy.

This paper is organized as follows. Section 2 introduces the generalized fractional oscillator process as a solution of a stochastic differential equation with two fractional orders. Despite that the covariance of this process does not have a closed analytic form, its basic properties can be studied by considering the asymptotic properties of its covariance. In particular, the fractal dimension and short-range dependence are studied. In section 4, we interpret the fractional oscillator process as a velocity process and study the long-time behavior of the corresponding position process. The fluctuation–dissipation relation is derived and the possibility of using the process to model anomalous diffusion is discussed. In section 5, we consider the fractional oscillator process as a one-dimensional Euclidean fractional scalar field. Stochastic quantization of the field at zero and finite temperatures is carried out. We proceed to calculate the Casimir energy associated with the fractional field at finite temperature by employing the zeta function regularization technique. In the last section, we briefly discuss possible applications and generalizations of the results obtained.

2. Fractional oscillator process with two indices

In this section we define the fractional oscillator process with two indices. Recall that an ordinary oscillator process $X(t)$ can be obtained as a solution to the Langevin equation

$$(D_t + \lambda)X(t) = \eta(t), \tag{2.1}$$

where $\eta(t) = \mathrm{d}B(t)/\mathrm{d}t$ is the standard white noise with

$$\langle \eta(t) \rangle = 0, \qquad \langle \eta(t)\eta(t') \rangle = \delta(t - t'),$$

$B(t)$ is the standard Brownian motion and λ is a positive constant. By using Fourier transform, the solution of (2.1) can be written as

$$X(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{\mathrm{e}^{\mathrm{i}\omega t}\hat{\eta}(\omega)}{\mathrm{i}\omega + \lambda} \, \mathrm{d}\omega,$$

where $\hat{\eta}(\omega)$ is the Fourier transform of the standard white noise $\eta(t)$:

$$\hat{\eta}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}t\omega} \, \mathrm{d}B(t).$$

In the literature, $X(t)$ is known as the oscillator process or the Ornstein–Uhlenbeck process. It is a centered stationary Gaussian process with the covariance function

$$\langle X(s)X(s+t) \rangle = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{\mathrm{e}^{\mathrm{i}\omega t}}{\omega^2 + \lambda^2} \, \mathrm{d}\omega = \frac{\mathrm{e}^{-\lambda|t|}}{2\lambda}. \tag{2.2}$$

One can also regard the oscillator process $X(t)$ as a one-dimensional Euclidean scalar Klein–Gordon field with mass λ and propagator given by the spectral density

$$S(\omega) = \frac{1}{2\pi} \frac{1}{\omega^2 + \lambda^2}.$$

Since fractal dynamics have increasingly played an important role in various transport phenomena in complex media, it would be interesting to investigate various possible generalizations of $X(t)$ to its fractional counterpart. One direct way is to replace the differential

operator D_t in (2.1) by the fractional differential operator $_aD_t$ to obtain the following type-I fractional Langevin equation [3, 15, 33, 34]:

$$\left(_aD_t^\alpha + \lambda\right)\mathsf{X}_{\alpha,1}(t) = \eta(t), \tag{2.3}$$

where the fractional derivative $_aD_t$ is defined as [2, 31, 35]:

$$\left(_aD_t^\alpha f\right)(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_a^t \frac{f(u)}{(t-u)^{\alpha-n+1}}\,\mathrm{d}u, \qquad n-1 < \alpha < n. $$

When $a = 0$, $_0D_t^\alpha$ is known as the Riemann–Liouville fractional derivative and for $a = -\infty$, $_{-\infty}D_t^\alpha$ is called the Weyl fractional derivative. A similar process with the Caputo fractional derivative [2, 31, 35] was considered in [36] and was also called the fractional oscillator. In [37–42], another class of fractional oscillators was defined by using fractional derivatives or fractional integrations, but with the noise $\eta(t)$ replaced by a deterministic driving force $F(t)$.

A less considered generalization of (2.1) is to fractionalize the operator $(_aD_t + \lambda)$ to obtain the following type-II fractional Langevin equation [16, 17]:

$$(_aD_t + \lambda)^\gamma \mathsf{X}_{1,\gamma}(t) = \eta(t), \qquad \gamma > 0. \tag{2.4}$$

The shifted fractional derivative $(_aD_t + \lambda)^\gamma$ can be informally defined as [43, 44]

$$(_aD_t + \lambda)^\gamma = \sum_{j=0}^\infty \binom{\gamma}{j} \lambda^j \, _aD_t^{\alpha(\gamma-j)}. \tag{2.5}$$

A more rigorous treatment can be carried out by using hypersingular integrals [45]. Gay and Heyde [43, 44] used (2.5) with $\alpha = 1$, $\gamma > 0$ in the study of a certain class of random fields. It is also used in defining the fractional Klein–Gordon scalar massive field $(-\Delta + m^2)^\gamma \phi(t) = 0$ [46, 47]. Compared to the process $\mathsf{X}_{\alpha,1}(t)$ (2.3), the process $\mathsf{X}_{1,\gamma}(t)$ (2.4) has the advantage that its covariance function has a closed form.

Recently, we have combined the investigation on these two types of processes and studied the more general case [30]:

$$\left(_aD_t^\alpha + \lambda\right)^\gamma \mathsf{X}_{\alpha,\gamma}(t) = \eta(t). \tag{2.6}$$

If the Weyl fractional derivative is used in (2.6), then for $\alpha\gamma > 1/2$, $\mathsf{X}_{\alpha,\gamma}(t)$ turns out to be a centered stationary Gaussian process with a representation

$$\mathsf{X}_{\alpha,\gamma}(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{\mathrm{e}^{i\omega t}\hat{\eta}(\omega)}{((i\omega)^\alpha + \lambda)^\gamma}\,\mathrm{d}\omega, $$

and the covariance function

$$\langle \mathsf{X}_{\alpha,\gamma}(s)\mathsf{X}_{\alpha,\gamma}(s+t)\rangle = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{\mathrm{e}^{i\omega t}}{|(i\omega)^\alpha + \lambda|^{2\gamma}}\,\mathrm{d}\omega = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{\mathrm{e}^{i\omega t}}{\left(|\omega|^{2\alpha} + 2\lambda|\omega|\cos\frac{\pi\alpha}{2} + \lambda^2\right)^\gamma}\,\mathrm{d}\omega. \tag{2.7}$$

The properties of the process $\mathsf{X}_{\alpha,\gamma}(t)$ have been studied in [30]. However, the above generalization contains an unsatisfactory aspect, namely its spectral density

$$S(\omega) = \frac{1}{2\pi} \frac{1}{\left(|\omega|^{2\alpha} + 2\lambda|\omega|\cos\frac{\pi\alpha}{2} + \lambda^2\right)^\gamma} \tag{2.8}$$

has a complicated form. When $\alpha = 1$, the spectral density simplifies to

$$S(\omega) = \frac{1}{2\pi} \frac{1}{(|\omega|^2 + \lambda^2)^\gamma}. \tag{2.9}$$

From the perspective of physical modeling, it is desirable to study a centered stationary Gaussian process $X_{\alpha,\gamma}(t)$ indexed by two parameters α, γ with $\alpha \in (0, 1]$, $\gamma > 0$, and with spectral density given by the simpler form

$$S_{\alpha,\gamma}(\omega) = \frac{1}{2\pi} \frac{1}{(|\omega|^{2\alpha} + \lambda^2)^\gamma} \tag{2.10}$$

compared to (2.8), which reduces to (2.9) when $\alpha = 1$. The spectral density (2.10) shows that the covariance function $C_{\alpha,\gamma}(t) = \langle X_{\alpha,\gamma}(s + t) X_{\alpha,\gamma}(s) \rangle$ of $X_{\alpha,\gamma}(t)$ is

$$C_{\alpha,\gamma}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{it\omega}}{(|\omega|^{2\alpha} + \lambda^2)^\gamma} \, d\omega, \tag{2.11}$$

and $X_{\alpha,\gamma}(t)$ has a representation given by

$$X_{\alpha,\gamma}(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{e^{i\omega t} \hat{\eta}(\omega)}{(|\omega|^{2\alpha} + \lambda^2)^{\frac{\gamma}{2}}} \, d\omega \tag{2.12}$$

when $\alpha\gamma > 1/2$. This later formula (2.12) signifies that $X_{\alpha,\gamma}(t)$ is a solution to the following fractional stochastic differential equation:

$$[(-\Delta)^\alpha + \lambda^2]^{\frac{\gamma}{2}} X_{\alpha,\gamma}(t) = \eta(t),$$

or equivalently, the equation

$$\left(\mathbf{D}_t^{2\alpha} + \lambda^2\right)^{\frac{\gamma}{2}} X_{\alpha,\gamma}(t) = \eta(t), \tag{2.13}$$

where $\mathbf{D}_t^{2\alpha}$ is the Riesz derivative defined by [31, 32]

$$\mathbf{D}_t^{2\alpha} f := (-\Delta)^\alpha f := F^{-1}(|\omega|^{2\alpha} \hat{f}(\omega)),$$

with $\hat{f} := F(f)$ the Fourier transform of f. The fractional operator $[(-\Delta)^\alpha + \lambda^2]^{\frac{\gamma}{2}}$ can be linked to the Bessel potential [45]. To avoid introducing too many different terminologies, we also call $X_{\alpha,\gamma}(t)$ a fractional oscillator process. It can also be known as a fractional Ornstein–Uhlenbeck process. Some simulations of the process $X_{\alpha,\gamma}(t)$ are given in figure 1.

Here we would like to remark that when $\alpha\gamma \leqslant 1/2$, $X_{\alpha,\gamma}(t)$ only exists as a generalized stochastic process, in the sense of generalized functions. Namely, for a Schwarz class test function $f(t)$,

$$\langle X_{\alpha,\gamma}(t), f(t) \rangle = \int_{\mathbb{R}} \frac{\hat{f}(-\omega) \hat{\eta}(\omega)}{(|\omega|^{2\alpha} + \lambda^2)^{\frac{\gamma}{2}}} \, d\omega,$$

where $\hat{f}(\omega)$ is the Fourier transform of f.

We would also like to remark that when $\alpha \in (0, 1]$, $\gamma > 0$ and $\lambda = 1$, the function $2\pi S_{\alpha,\gamma}(t)$ appears as the covariance function of the Gaussian process with generalized Cauchy covariance [21]. Therefore, the function $C_{\alpha,\gamma}(t)$ is positive. This implies that the random process $X_{\alpha,\gamma}(t)$ is a persistent process.

The two fractional generalizations $\mathsf{X}_{\alpha,\gamma}(t)$ and $X_{\alpha,\gamma}(t)$ of an ordinary oscillator process do not give the same solutions since they give rise to different spectral densities. However, if we would like to regard the fractional oscillator process as a one-dimensional fractional Euclidean Klein–Gordon field, (2.13) is a one-dimensional Klein–Gordon equation with two fractional orders, and the covariance function (2.11) becomes the propagator of the corresponding fractional Klein–Gordon field $\phi_{\alpha,\gamma}(t)$. We shall show in section 5 that $\phi_{\alpha,\gamma}(t)$ can be obtained by applying Parisi–Wu stochastic quantization [48] involving a nonlocal Euclidean action.
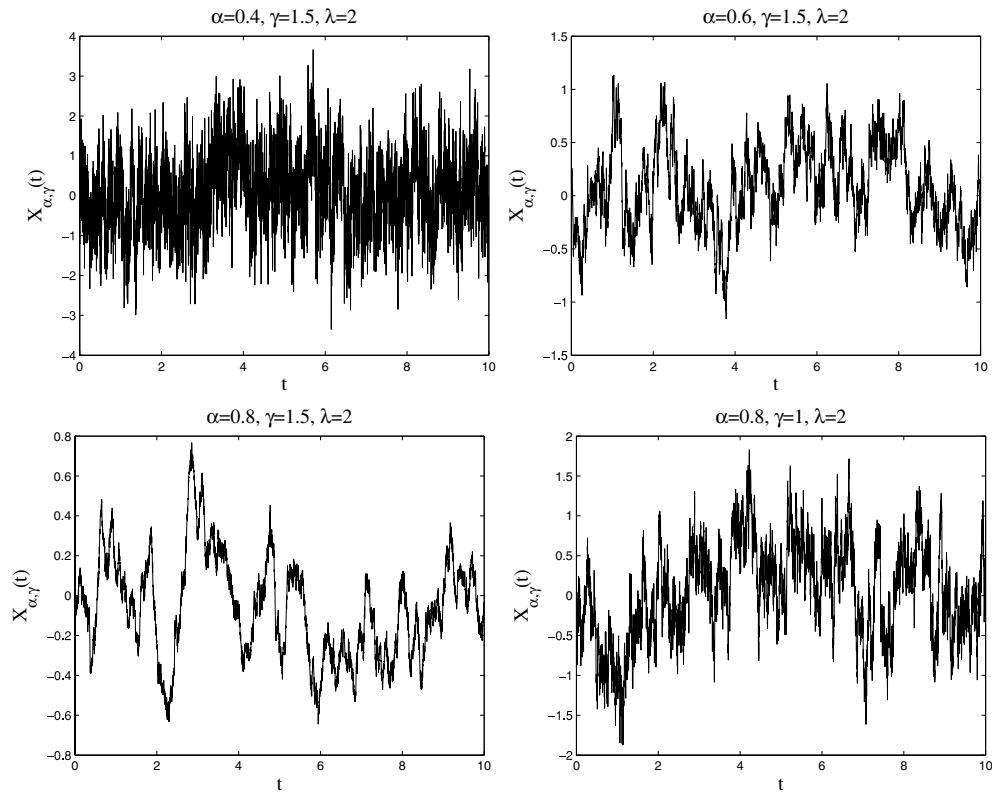
Figure 1. Some simulations of $X_{\alpha,\gamma}(t)$ for different values of α and γ.

3. Asymptotic behaviors of the covariance function $C_{\alpha,\gamma}(t)$ and sample path properties of $X_{\alpha,\gamma}(t)$

The study of sample path properties of a stochastic process such as its local regularity and long-time behavior is important for determining the parameters when we want to apply the process for modeling. In this section, we first derive the asymptotic behavior of the covariance function $C_{\alpha,\gamma}(t)$ at small and large time, and then apply the results to derive the local regularity of the sample paths of $X_{\alpha,\gamma}(t)$ and its short memory property.

3.1. Asymptotic behaviors of the covariance function $C_{\alpha,\gamma}(t)$

When $\alpha = 1, \gamma > 0$, the covariance function $C_{1,\gamma}(t)$ (2.11) has the following closed form [16]:

$$C_{1,\gamma}(t) = \frac{2^{1/2-\gamma}}{\sqrt{\pi}\,\Gamma(\gamma)} \left(\frac{|t|}{\lambda}\right)^{\gamma-1/2} K_{\gamma-1/2}(\lambda|t|), \tag{3.1}$$

where $K_\nu(z)$ is the modified Bessel function of second kind or the MacDonald function. However, the covariance function $C_{\alpha,\gamma}(t)$ in general does not exist in a closed analytic form. In fact, since the spectral density (2.10) has the same functional form as the characteristic function of the generalized Linnik distribution [49], the covariance function $C_{\alpha,\gamma}(t)$ (2.11) has

the same functional form as the Linnik probability density function, whose analytic properties have been studied in [49, 50]. In particular, one can obtain the following integral representation for the covariance function $C_{\alpha,\gamma}(t)$:

$$C_{\alpha,\gamma}(t) = \frac{1}{\pi} \operatorname{Im} \int_0^\infty \frac{\mathrm{e}^{-u|t|}\,\mathrm{d}u}{(\lambda^2 + \mathrm{e}^{-\mathrm{i}\pi\alpha}u^{2\alpha})^\gamma}. \tag{3.2}$$

It turns out that the analytic properties of the Linnik probability density function depend on the arithmetic nature of the parameters α and γ; and the conditions imposed on α and γ are rather complicated and are not of practical interest. Therefore, we shall use different methods for studying the asymptotic behavior of the covariance function $C_{\alpha,\gamma}(t)$ that are more suitable for various applications.

For the properties of $X_{\alpha,\gamma}(t)$ that we are interested in, such as its fractal dimension, long or short-range dependence, it suffices for us to know the leading behavior of the variance of the associated increment process

$$\sigma_{\alpha,\gamma}^2(t) = \langle [X_{\alpha,\gamma}(s+t) - X_{\alpha,\gamma}(s)]^2 \rangle = 2C_{\alpha,\gamma}(0) - 2C_{\alpha,\gamma}(t) \tag{3.3}$$

for $t \to 0$ and the leading behavior of $C_{\alpha,\gamma}(t)$ for $t \to \infty$.

First, we examine the behavior of $\sigma_{\alpha,\gamma}^2(t)$ when $t \to 0$. We impose the restriction $\alpha\gamma > 1/2$ so that $X_{\alpha,\gamma}(t)$ has finite variance and (3.3) is well defined. Under this restriction, the variance $C_{\alpha,\gamma}(0)$ is given by (#3.251, no.11, [51])

$$\begin{aligned}
C_{\alpha,\gamma}(0) &= \frac{1}{\pi} \int_0^\infty \frac{\mathrm{d}\omega}{(\omega^{2\alpha} + \lambda^2)^\gamma} \\
&= \frac{1}{2\pi\alpha} \frac{\Gamma\left(\frac{1}{2\alpha}\right)\Gamma\left(\gamma - \frac{1}{2\alpha}\right)}{\Gamma(\gamma)} \lambda^{\frac{1}{\alpha}-2\gamma} \\
&= \frac{\lambda^{\frac{1}{\alpha}-2\gamma}}{2\alpha\Gamma(\gamma)} \frac{\Gamma\left(\frac{1}{2\alpha}\right)}{\Gamma\left(1 - \gamma + \frac{1}{2\alpha}\right)\sin\left(\pi\left(\gamma - \frac{1}{2\alpha}\right)\right)},
\end{aligned} \tag{3.4}$$

where the identity $-z\Gamma(z)\Gamma(-z) = \pi/\sin(\pi z)$ has been used. Our result is in agreement with that of Erdogan and Ostrovskii [49]. On the other hand, when $\alpha\gamma < 1/2$, the covariance of the generalized Gaussian process $X_{\alpha,\gamma}(t)$ satisfies

$$\begin{aligned}
\langle X_{\alpha,\gamma}(t)X_{\alpha,\gamma}(t') \rangle = C_{\alpha,\gamma}(|t - t'|) &= \frac{1}{\pi} \int_0^\infty \frac{\cos(\omega|t-t'|)}{(\omega^{2\alpha} + \lambda^2)^\gamma}\,\mathrm{d}\omega \\
&= \frac{|t - t'|^{2\alpha\gamma-1}}{\pi} \int_0^\infty \frac{\cos\omega}{(\omega^{2\alpha} + \lambda^2|t-t'|^{2\alpha})^\gamma}\,\mathrm{d}\omega \\
&= \frac{|t - t'|^{2\alpha\gamma-1}}{\pi} \int_0^\infty \frac{\cos\omega}{\omega^{2\alpha\gamma}}\,\mathrm{d}\omega + o(|t - t'|^{2\alpha\gamma-1}) \\
&= \Gamma(1 - 2\alpha\gamma)\frac{\sin(\pi\alpha\gamma)}{\pi}|t - t'|^{2\alpha\gamma-1} + o(|t - t'|^{2\alpha\gamma-1})
\end{aligned}$$

as $|t - t'| \to 0$. We have used #3.761, no. 9 in [51].

Now we return to the case of ordinary fields. To obtain the leading behavior of

$$\sigma_{\alpha,\gamma}^2(t) = 2C_{\alpha,\gamma}(0) - 2C_{\alpha,\gamma}(t) = \frac{2}{\pi} \int_0^\infty \frac{1 - \cos(\omega|t|)}{(\lambda^2 + \omega^{2\alpha})^\gamma}\,\mathrm{d}\omega, \tag{3.5}$$

we consider the cases $1/2 < \alpha\gamma < 3/2$, $\alpha\gamma > 3/2$ and $\alpha\gamma = 3/2$ separately.

Case I. When $1/2 < \alpha\gamma < 3/2$, (3.5) is equal to

$$\sigma_{\alpha,\gamma}^2(t) = \frac{4}{\pi} \int_0^\infty \frac{\sin^2\left(\frac{\omega|t|}{2}\right)}{(\lambda^2 + \omega^{2\alpha})^\gamma} \, d\omega$$

$$= \frac{4|t|^{2\alpha\gamma-1}}{\pi} \int_0^\infty \frac{\sin^2(\omega/2)}{(\omega^{2\alpha} + \lambda^2|t|^{2\alpha})^\gamma} \, d\omega$$

$$= \frac{4|t|^{2\alpha\gamma-1}}{\pi} \int_0^\infty \omega^{-2\alpha\gamma} \sin^2\left(\frac{\omega}{2}\right) d\omega + o(|t|^{2\alpha\gamma-1})$$

$$= -\frac{|t|^{2\alpha\gamma-1}}{\cos(\pi\alpha\gamma)\Gamma(2\alpha\gamma)} + o(|t|^{2\alpha\gamma-1}) \qquad \text{as} \quad t \to 0. \tag{3.6}$$

We have used #3.823 of [51] in the last step. Equation (3.6) shows that when $1/2 < \alpha\gamma < 3/2$ and $t \to 0$, the leading term of $\sigma_{\alpha,\gamma}^2(t)$ is of order $|t|^{2\alpha\gamma-1}$. If we replace $\alpha\gamma$ by $H + 1/2$, then (3.6) becomes

$$\sigma_{\alpha,\gamma}^2(t) \sim \frac{|t|^{2H}}{\sin(\pi H)\Gamma(2H+1)} + o(|t|^{2H}) \qquad \text{as} \quad t \to 0,$$

which shows that the short-time asymptotic behavior of $\sigma_{\alpha,\gamma}^2(t)$ is characterized by the index $H = \alpha\gamma - 1/2$.

Case II. For $\alpha\gamma > 3/2$, using $1 - \cos(\omega|t|) = \frac{1}{2}\omega^2|t|^2 + O(|t|^4)$ as $t \to 0$, we have

$$\sigma_{\alpha,\gamma}^2(t) = \frac{|t|^2}{\pi} \int_0^\infty \frac{\omega^2 \, d\omega}{(\lambda^2 + \omega^{2\alpha})^\gamma} + o(|t|^2)$$

$$= \frac{|t|^2}{2\pi\alpha} \lambda^{\frac{3}{\alpha}-2\gamma} \frac{\Gamma\left(\frac{3}{2\alpha}\right)\Gamma\left(\gamma - \frac{3}{2\alpha}\right)}{\Gamma(\gamma)} + o(|t|^2). \tag{3.7}$$

This shows that after crossing the point $\alpha\gamma = 3/2$, the leading behavior of $\sigma_{\alpha,\gamma}^2(t)$ is of order $|t|^2$, which does not depend on the parameters α and γ.

Case III. The borderline case $\alpha\gamma = 3/2$ is more complicated. First, we find as in case I that

$$\sigma_{\alpha,\gamma}^2(t) = \frac{4|t|^2}{\pi} \int_0^\infty \frac{\sin^2(\omega/2)}{(\omega^{2\alpha} + \lambda^2|t|^{2\alpha})^\gamma} \, d\omega.$$

The integral

$$\int_0^\infty \frac{\sin^2(\omega/2)}{(\omega^{2\alpha} + \lambda^2|t|^{2\alpha})^\gamma} \, d\omega \tag{3.8}$$

does not converge when $t \to 0$ because of the singularity at the origin of the integrand $\omega^{-2\alpha\gamma} \sin^2(\omega/2)$. Since $\sin z \sim z$ when $z \to 0$, we write (3.8) as a sum of two terms $A_1(t)$ and $A_2(t)$, where

$$A_1(t) = \int_0^1 \frac{(\omega/2)^2}{(\omega^{2\alpha} + \lambda^2|t|^{2\alpha})^\gamma} \, d\omega$$

reflects the divergence of (3.8) when $t \to 0$; and

$$A_2(t) = \int_0^1 \frac{\sin^2(\omega/2) - (\omega/2)^2}{(\omega^{2\alpha} + \lambda^2|t|^{2\alpha})^\gamma} \, d\omega + \int_1^\infty \frac{\sin^2(\omega/2)}{(\omega^{2\alpha} + \lambda^2|t|^{2\alpha})^\gamma} \, d\omega$$

carries the finite part. By making a change of variable $v = \omega^{2\alpha}$ or equivalently $\omega = v^{\gamma/3}$, we find that

$$A_1(t) = \frac{1}{8\alpha} \int_0^1 \frac{v^{\gamma-1} \, dv}{(v + \lambda^2|t|^{2\alpha})^\gamma}.$$

To find the asymptotic behavior of $A_1(t)$ as $t \to 0$, we split it again as the sum of $A_3(t)$ and $A_4(t)$, where

$$A_3(t) = \frac{1}{8\alpha} \int_0^1 \frac{dv}{(v + \lambda^2 |t|^{2\alpha})} \sim \frac{1}{4} \log \frac{1}{|t|} - \frac{1}{4\alpha} \log \lambda + o(1) \qquad (3.9)$$

gives the divergence part; and $A_4(t) := A_1(t) - A_3(t)$ gives a finite limit:

$$A_4(t) = \frac{1}{8\alpha} \int_0^1 \frac{v^{\gamma-1} - (v + \lambda^2 |t|^{2\alpha})^{\gamma-1}}{(v + \lambda^2 |t|^{2\alpha})^\gamma} \, dv = \frac{1}{8\alpha} \int_0^{\frac{1}{\lambda^2 |t|^{2\alpha}}} \frac{v^{\gamma-1} - (1 + v)^{\gamma-1}}{(1 + v)^\gamma} \, dv$$

$$\sim \frac{1}{8\alpha} \int_0^\infty \frac{v^{\gamma-1} - (1 + v)^{\gamma-1}}{(1 + v)^\gamma} \, dv + o(1) = \frac{1}{8\alpha}(\psi(1) - \psi(\gamma)) + o(1). \qquad (3.10)$$

In the last equality, we have used #3.219 of [51], with $\psi(z) = \Gamma'(z)/\Gamma(z)$ being the logarithmic derivative of the gamma function. The limit of $A_2(t)$ when $t \to 0$ is given by

$$A_2(0) = \int_0^1 \omega^{-3} \left(\sin^2 \left(\frac{\omega}{2} \right) - \left(\frac{\omega}{2} \right)^2 \right) d\omega + \int_1^\infty \omega^{-3} \sin^2 \left(\frac{\omega}{2} \right) d\omega.$$

It can be computed explicitly by the regularization method:

$$A_2(0) = \lim_{\varepsilon \to 0^+} \left\{ \int_0^\infty \omega^{-3+\varepsilon} \sin^2 \left(\frac{\omega}{2} \right) d\omega - \frac{1}{4} \int_0^1 \omega^{-1+\varepsilon} \, d\omega \right\}$$

$$= \lim_{\varepsilon \to 0^+} \left\{ \frac{1}{2} \cos \frac{\pi \varepsilon}{2} \frac{\Gamma(1 + \varepsilon)}{\varepsilon(1 - \varepsilon)(2 - \varepsilon)} - \frac{1}{4\varepsilon} \right\} = \frac{1}{4} \left(\psi(1) + \frac{3}{2} \right). \qquad (3.11)$$

Combining (3.9), (3.10) and (3.11), we find that the leading behavior of $\sigma_{\alpha,\gamma}^2(t)$ when $\alpha\gamma = 3/2$ is

$$\sigma_{\alpha,\gamma}^2(t) \sim \frac{|t|^2}{\pi} \log \frac{1}{|t|} - \frac{|t|^2}{\pi} \left\{ \frac{1}{\alpha} \log \lambda + \frac{1}{2\alpha}(\psi(\gamma) - \psi(1)) - \psi(1) - \frac{3}{2} \right\} + o(1). \qquad (3.12)$$

This shows that the leading behavior of $\sigma_{\alpha,\gamma}^2(t)$ in the borderline case $\alpha\gamma = 3/2$ is of order $|t|^2 \log(1/|t|)$.

The behavior of $\sigma_{\alpha,\gamma}^2(t)$ at small t is illustrated graphically in figures 2 and 3. We can further confirm our results by checking the small time behavior of $\sigma_{\alpha,\gamma}^2(t)$ for the case $\alpha = 1$, with the covariance function $C_{1,\gamma}(t)$ given explicitly by (3.1). By using #8.445, 8.446 and 8.485 of [51] one gets for $\nu \notin \mathbb{Z}$,

$$K_\nu(z) = K_{-\nu}(z) = \frac{\pi}{2 \sin(\pi\nu)} \left\{ \sum_{j=0}^\infty \frac{(z/2)^{2j-\nu}}{j! \Gamma(j+1-\nu)} - \sum_{j=0}^\infty \frac{(z/2)^{2j+\nu}}{j! \Gamma(j+1+\nu)} \right\} \qquad (3.13)$$
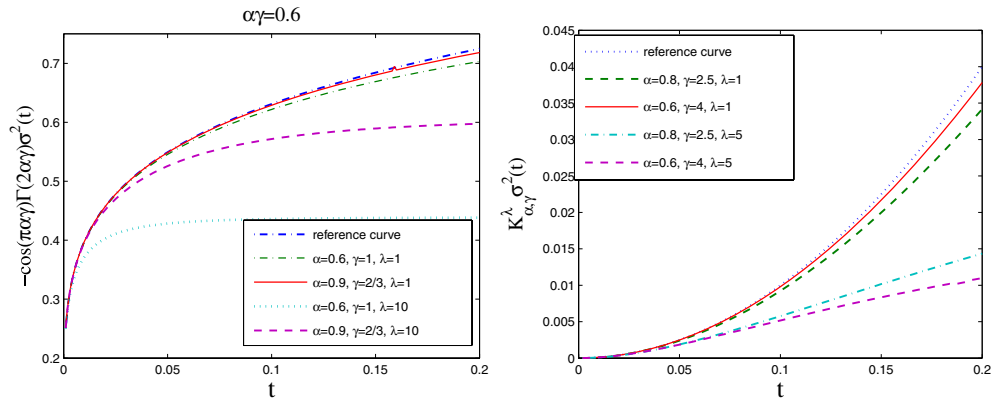
and in the case $\nu = \pm m$, m a non-negative integer,

$$K_\nu(z) = \frac{1}{2} \sum_{j=0}^{m-1} \frac{(-1)^j (m-j-1)!}{j!} \left(\frac{z}{2} \right)^{2j-m}$$

$$+ (-1)^{m+1} \sum_{j=0}^\infty \frac{(z/2)^{m+2j}}{j!(m+j)!} \left\{ \log \frac{z}{2} - \frac{1}{2}\psi(j+1) - \frac{1}{2}\psi(j+1+m) \right\}. \qquad (3.14)$$

From (3.13) and (3.14), one finds that the variance of $X_{1,\gamma}(t)$ is

$$C_{1,\gamma}(0) = \frac{\sqrt{\pi}}{2\lambda^{2\gamma-1} \sin\left(\pi\left(\gamma - \frac{1}{2}\right)\right)\Gamma(\gamma)\Gamma\left(\frac{3}{2} - \gamma\right)}$$

and the leading behavior of $\sigma_{1,\gamma}^2(t)$ as $t \to 0$ is given by

Figure 2. These graphs show the small-time behavior of $\sigma_{\alpha,\gamma}^2(t)$. Left: the function $-\cos(\pi\alpha\gamma)\Gamma(2\alpha\gamma)\sigma_{\alpha,\gamma}^2(t)$ is plotted as a function of t. The reference curve is $y = |t|^{2\alpha\gamma-1}$. Here $\alpha\gamma = 0.6$. The graph shows that $-\cos(\pi\alpha\gamma)\Gamma(2\alpha\gamma)\sigma_{\alpha,\gamma}^2(t) \sim |t|^{2\alpha\gamma-1}$ when $t \to 0$. Right: the function $K_{\alpha,\gamma}^\lambda \sigma_{\alpha,\gamma}^2(t)$ is plotted as a function of t, where $K_{\alpha,\gamma}^\lambda = 2\pi\alpha\lambda^{2\gamma-\frac{3}{\alpha}} \frac{\Gamma(\gamma)}{\Gamma(\frac{3}{2\alpha})\Gamma(\gamma-\frac{3}{2\alpha})}$. The reference curve is $y = t^2$. The graph shows that $K_{\alpha,\gamma}^\lambda \sigma_{\alpha,\gamma}^2(t) \sim t^2$ when $t \to 0$.
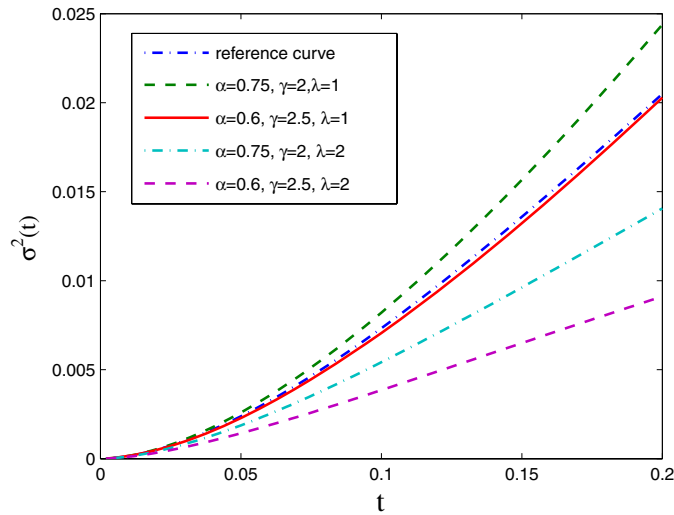
Figure 3. The graph shows the small time behavior of $\sigma_{\alpha,\gamma}^2(t)$ when $\alpha\gamma = 3/2$. The reference curve is $t^2 \log(1/|t|)$.

• If $1/2 < \gamma < 3/2$,

$$
\begin{aligned}
\sigma_{1,\gamma}^2(t) &\sim \frac{\sqrt{\pi}|t|^{2\gamma-1}}{2^{2\gamma-1}\sin\left(\pi\left(\gamma-\frac{1}{2}\right)\right)\Gamma(\gamma)\Gamma\left(\gamma-\frac{1}{2}\right)} \\
&= -\frac{|t|^{2\gamma-1}}{\cos(\pi\gamma)\Gamma(2\gamma)}.
\end{aligned}
\tag{3.15}
$$

In the last step, we have used the identity $\Gamma(2z) = (2^{2z-1}/\sqrt{\pi})\Gamma(z)\Gamma(z+(1/2))$.

• If $\gamma > 3/2$,

$$\sigma_{1,\gamma}^2(t) \sim \frac{1}{4\sqrt{\pi}} \frac{\Gamma\left(\gamma - \frac{3}{2}\right)}{\Gamma(\gamma)} |t|^2. \tag{3.16}$$

• If $\gamma = 3/2$,

$$\sigma_{1,\gamma}^2(t) \sim \frac{|t|^2}{\pi} \left(\log \frac{1}{|t|} - \log \lambda + \log 2 + \psi(1) + \frac{1}{2} \right). \tag{3.17}$$

By putting $\alpha = 1$ in the small time asymptotic formulae (3.6), (3.7) and (3.12) of $\sigma_{1,\gamma}^2(t)$, one obtains formulae (3.15)–(3.17). This corroborates the results (3.6), (3.7) and (3.12).

Next, we study the asymptotic behavior of $C_{\alpha,\gamma}(t)$ for $t \to \infty$. When $0 < \alpha < 1$, we can make use of (3.2). Making a change of variable $u \mapsto u/t$ and using the formula

$$\frac{1}{(1+z)^\gamma} = \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(\gamma + j)}{j! \Gamma(\gamma)} z^j,$$

we can derive the following $t \to \infty$ asymptotic expression for $C_{\alpha,\gamma}(t)$ which is valid when $\alpha \in (0, 1)$:

$$
\begin{aligned}
C_{\alpha,\gamma}(t) &= \frac{1}{\pi} \mathrm{Im} \left\{ t^{-1} \int_0^\infty \frac{\mathrm{e}^{-u}\, \mathrm{d}u}{\left(\lambda^2 + \mathrm{e}^{-\mathrm{i}\pi\alpha} \frac{u^{2\alpha}}{t^{2\alpha}}\right)^\gamma} \right\} \\
&\sim \frac{1}{\pi} \mathrm{Im} \left\{ t^{-1} \int_0^\infty \mathrm{e}^{-u} \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(\gamma + j)}{j! \Gamma(\gamma)} \mathrm{e}^{-\mathrm{i}\pi\alpha j} \frac{u^{2\alpha j}}{t^{2\alpha j}} \lambda^{-2\gamma - 2j} \right\} \\
&\sim \frac{1}{\pi \Gamma(\gamma)} \sum_{j=1}^{\infty} \frac{(-1)^{j+1} \lambda^{-2(\gamma+j)}}{j!} \Gamma(\gamma + j) \Gamma(1 + 2\alpha j) \sin(\pi\alpha j) t^{-(2\alpha j + 1)}.
\end{aligned} \tag{3.18}
$$

The leading term is

$$C_{\alpha,\gamma}(t) \sim \frac{\gamma}{\pi} \lambda^{-2(\gamma+1)} \Gamma(1 + 2\alpha) \sin(\pi\alpha) t^{-(2\alpha+1)}. \tag{3.19}$$

By letting $\lambda = 1$, (3.18) is in agreement with the analogous result given in [49] for Linnik distribution. Note that when $0 < \alpha < 1$, $C_{\alpha,\gamma}(t)$ is of polynomial decay with order $t^{-2\alpha-1}$ when $t \to \infty$ (see figures 4 and 5).

The case $\alpha = 1$ has to be considered separately. Using the relation (#8.451, no. 6, [51]):

$$K_\nu(z) = \sqrt{\frac{\pi}{2z}} \mathrm{e}^{-z} \left(1 + \frac{4\nu^2 - 1}{8z} + \cdots \right),$$

we find from the explicit formula (3.1) for $C_{1,\gamma}(t)$ that

$$C_{1,\gamma}(t) \sim \frac{|t|^{\gamma-1}}{(2\lambda)^\gamma \Gamma(\gamma)} \mathrm{e}^{-\lambda|t|} \qquad \text{as} \quad |t| \to \infty. \tag{3.20}$$

One notes that at large time, $C_{1,\gamma}(t)$ decays exponentially, in contrast to the large time behavior of $C_{\alpha,\gamma}(t)$, $\alpha \in (0, 1)$ (3.18), which decays polynomially (see figure 5).

From the above results, it appears that the small time asymptotic behavior of the variance $\sigma_{\alpha,\gamma}^2(t)$ varies as $|t|^{\min\{2\alpha\gamma-1,2\}}$, depending on both α and γ. However, if the index γ is replaced by γ/α, then $\sigma_{\alpha,\gamma}^2(t) \sim |t|^{\min\{2\gamma-1,2\}}$ as $t \to 0$. Thus, together with the large time asymptotic behavior $C_{\alpha,\gamma}(t) \sim |t|^{-2\alpha-1}$ as $t \to \infty$, we have the result that the small and large time asymptotic behaviors of the covariance of $X_{\alpha,\gamma}(t)$ are separately characterized by γ and α. The physical implications of this result will be discussed in the subsequent sections.
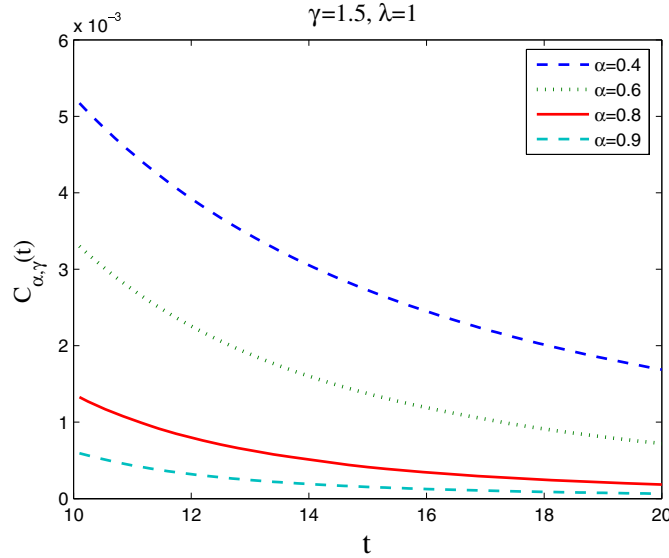
Figure 4. This graph shows the large time behavior of $C_{\alpha,\gamma}(t)$.
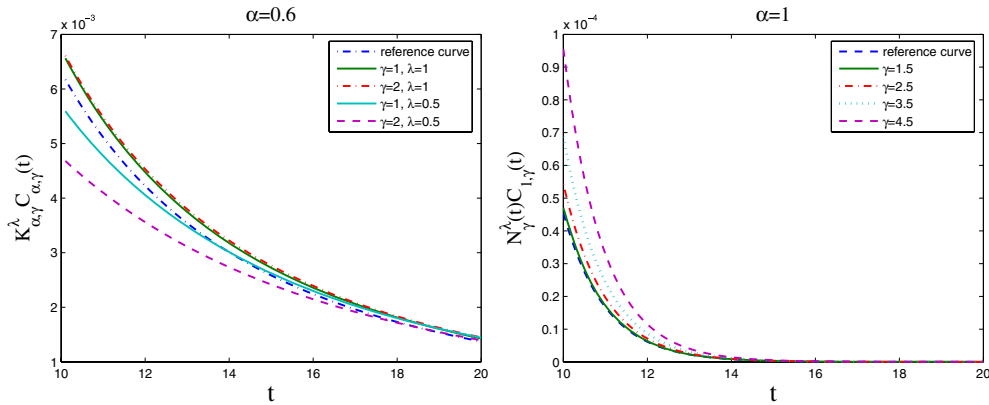

Figure 5. The large time behavior of $C(t)$ when $\alpha = 0.6$ and $\alpha = 1$ respectively. Left: here $K_{\alpha,\gamma}^{\lambda} = \frac{\lambda^{2(\gamma+1)}\pi}{\gamma\Gamma(1+2\alpha)\sin(\pi\alpha)}$ and the reference curve is $y = t^{-(2\alpha+1)}$. The graph shows that $K_{\alpha,\gamma}^{\lambda} C_{\alpha,\gamma}(t) \sim t^{-2\alpha-1}$ as $t \to \infty$. Right: here $N_{\gamma}^{\lambda}(t) = (2\lambda)^{\gamma}\Gamma(\gamma)t^{\gamma-1}$, $\lambda = 1$ and the reference curve is $y = e^{-\lambda t}$. The graph shows that $N_{\gamma}^{\lambda}(t)C_{1,\gamma}(t) \sim e^{-\lambda t}$ for large t.

3.2. Locally asymptotically self-similarity and fractal dimension

Recall that the stationary random process cannot be self-similar [11]. It would be interesting to see whether $X_{\alpha,\gamma}(t)$ satisfies a weaker self-similar property, namely self-similarity at very small timescales. First we introduce some definitions. A positive function f is asymptotically homogeneous of order κ at ∞ if there exists a non-zero function f^{∞} such that, for almost every $\omega \in \mathbb{R}$ and $r > 0$, $f^r(\omega) = r^{-\kappa} f(r\omega)$ has a limit $f^{\infty}(\omega)$ when $r \to \infty$. Clearly, $f^{\infty}(\omega)$ is homogeneous of order κ, thus fixes the index κ uniquely. One can easily verify that the spectral density $S_{\alpha,\gamma}(\omega)$ is asymptotically homogeneous of order $2\alpha\gamma$ at ∞, with $S_{\alpha,\gamma}^{\infty}(\omega) = \omega^{-2\alpha\gamma}/(2\pi)$. In addition, the spectral density satisfies the following property:

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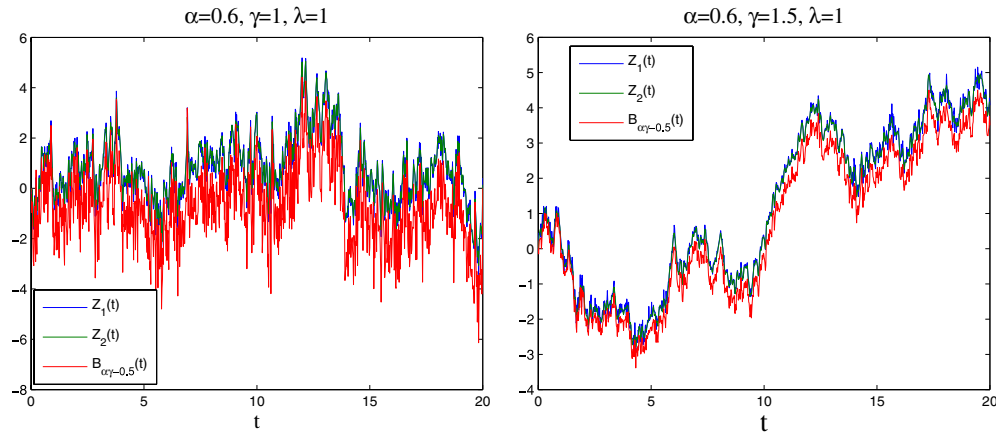
Figure 6. The graphs show that $\varepsilon^{-\alpha\gamma+\frac{1}{2}}[X_{\alpha,\gamma}(\varepsilon t) - X_{\alpha,\gamma}(0)]$ approaches the fractional Brownian motion $B_{\alpha\gamma-\frac{1}{2}}(t)$ when $\varepsilon \to 0$. Here $Z_i(t) = \varepsilon_i^{-\alpha\gamma+\frac{1}{2}}[X_{\alpha,\gamma}(\varepsilon_i t) - X_{\alpha,\gamma}(0)]$ with $\varepsilon_1 = 0.0001, \varepsilon_2 = 0.000\,05, i = 1, 2$.

there exist positive constants $A, B \in \mathbb{R}$ such that $S_{\alpha,\gamma}(\omega) \leqslant B|\omega|^{-2\alpha\gamma}$, for almost all $|\omega| > A$. This is clearly true since $\lambda > 0$ implies

$$S_{\alpha,\gamma}(\omega) = \frac{1}{2\pi} \frac{1}{(\lambda^2 + \omega^{2\alpha})^\gamma} < \frac{|\omega|^{-2\alpha\gamma}}{2\pi}.$$

Using a result (proposition 2 in [52]), one concludes that if $\alpha\gamma \in (1/2, 3/2)$, the fractional process $X_{\alpha,\gamma}(t)$ is locally asymptotically self-similar (LASS) of order $\alpha\gamma - 1/2$, that is for $u \in \mathbb{R}$,

$$\lim_{\varepsilon \to 0^+} \left\{ \frac{X_{\alpha,\gamma}(t_0 + \varepsilon u) - X_{\alpha,\gamma}(t_0)}{\varepsilon^{\alpha\gamma - \frac{1}{2}}} \right\} = \left\{ X_{\alpha,\gamma}^\infty(u) \right\},$$

where the convergence is in the sense of distribution. Note that this result is in agreement with (3.6) which asserts that the covariance $\sigma_{\alpha,\gamma}^2(t) \sim C|t|^{2\alpha\gamma-1}$ as $t \to 0$. The limit process or the tangent process $X_{\alpha,\gamma}^\infty$ is self-similar of order $\alpha\gamma - \frac{1}{2}$. It can be identified with fractional Brownian motion if $\alpha\gamma = H + 1/2$, where H denotes the Hurst index of the fractional Brownian motion. Just like the case of ordinary oscillator process, which locally behaves like Brownian motion, the fractional oscillator process $X_{\alpha,\gamma}(t)$ has the same local behavior as a fractional Brownian motion of order $\alpha\gamma - 1/2$. In fact, (3.6) gives us

$$\left\langle \lim_{\varepsilon \to 0^+} \left[\frac{X_{\alpha,\gamma}(t_0 + \varepsilon u) - X_{\alpha,\gamma}(t_0)}{\varepsilon^{\alpha\gamma - \frac{1}{2}}} \right] \left[\frac{X_{\alpha,\gamma}(t_0 + \varepsilon v) - X_{\alpha,\gamma}(t_0)}{\varepsilon^{\alpha\gamma - \frac{1}{2}}} \right] \right\rangle$$

$$= \lim_{\varepsilon \to 0} \frac{\sigma_{\alpha,\gamma}^2(\varepsilon u) + \sigma_{\alpha,\gamma}^2(\varepsilon v) - \sigma_{\alpha,\gamma}^2(\varepsilon(u - v))}{2\varepsilon^{2\alpha\gamma - 1}}$$

$$= -\frac{1}{2\cos(\pi\alpha\gamma)\Gamma(2\alpha\gamma)} (|u|^{2\alpha\gamma-1} + |v|^{2\alpha\gamma-1} - |u - v|^{2\alpha\gamma-1}),$$

which is the covariance function $\langle B_H(u)B_H(v) \rangle$ for the fractional Brownian motion

$$B_H(u) := \frac{1}{\sqrt{2\pi}} \int_\mathbb{R} \frac{e^{i\omega t} - 1}{|\omega|^{H+\frac{1}{2}}} \, d\omega,$$

if we identify H with $\alpha\gamma - 1/2$ (see figure 6). We also remark that when $\alpha\beta = 3/2$, we find from (3.12) that the fractional oscillator process $X_{\alpha,\gamma}(t)$ fails to satisfy the

LASS property. When $\alpha\gamma$ exceeds $3/2$, the process $X_{\alpha,\beta}(t)$ becomes differentiable with variance

$$\lim_{\varepsilon\to0}\left\langle\left[\frac{X_{\alpha,\gamma}(t_0+\varepsilon)-X_{\alpha,\gamma}(t_0)}{\varepsilon}\right]^2\right\rangle=\lim_{\varepsilon\to0}\frac{\sigma_{\alpha,\gamma}^2(\varepsilon)}{\varepsilon^2}=\frac{\lambda^{\frac{3}{\alpha}-2\gamma}}{2\pi\alpha}\frac{\Gamma\left(\frac{3}{2\alpha}\right)\Gamma\left(\gamma-\frac{3}{2\alpha}\right)}{\Gamma(\gamma)},$$

which follows from (3.7).

Another important concept in the study of the sample path properties of a stochastic process is the Hölderian property. A function $f:[a,b]\to\mathbb{R}$ is Hölderian of order $\kappa\in(0,1]$ if

$$|f(t)-f(s)|\leqslant K|t-s|^{\kappa},\qquad\text{for all}\quad s,t\in[a,b]$$

for some constant $K>0$. It is well known that if $Z(t)$ is a stationary process and $\sigma^2(t)=\langle[Z(t)-Z(0)]^2\rangle$ satisfies

$$\sigma^2(t)\leqslant C|t|^{2\kappa},$$

then almost surely (a.s.) the sample path of $Z(t)$ is Hölderian of order $\kappa-\varepsilon$ for all $\varepsilon>0$ [53–55]. Applying this concept to the fractional oscillator process $X_{\alpha,\gamma}(t)$, we find from (3.6), (3.7) and (3.12) that for any $\varepsilon>0$, the sample path of $X_{\alpha,\gamma}(t)$ is Hölderian of order $\alpha\gamma-1/2-\varepsilon$ if $\alpha\gamma<3/2$, and is Hölderian of order $1-\varepsilon$ if $\alpha\gamma\geqslant3/2$.

Now we would like to consider the fractal dimension D of the graph for the fractional oscillator process $X_{\alpha,\gamma}(t)$. Since fractal dimension is a local concept, fractality is defined for infinitesimally small timescales. For a locally self-similar process, one may apply the following result to obtain its fractal dimension. A process which is LASS of order $\kappa>0$ and its sample paths are a.s. $\kappa-\varepsilon$-Hölderian for all $\varepsilon>0$, then the fractal dimension of its graph is a.s. equal to $2-\kappa$ [53, 56]. Applying this result to $X_{\alpha,\gamma}(t)$ gives the fractal dimension $D=\frac{5}{2}-\alpha\gamma$ for the graph of the fractional oscillator process when $1/2<\alpha\gamma<3/2$. For $\alpha\gamma\geqslant3/2$, the fractal dimension of the graph of $X_{\alpha,\gamma}(t)$ is equal to 1. In other words, the fractal dimension of the graph of $X_{\alpha,\gamma}(t)$ is $\max\left\{1,\frac{5}{2}-\alpha\gamma\right\}$. Again, if we replace γ by γ/α, then the fractal dimension becomes $\max\left\{1,\frac{5}{2}-\gamma\right\}$, which depends solely on γ.

3.3. Short-range dependence property

First we recall that the Ornstein–Uhlenbeck process or the ordinary oscillator process (up to a multiplicative constant) is the only stationary Gaussian Markov process (see, e.g., [53]), thus we rule out the possibility of the fractional oscillator processes $X_{\alpha,\gamma}(t)$ being Markovian. Now we would like to find out the nature of memory possessed by $X_{\alpha,\gamma}(t)$, whether it has long memory or long-range dependence (LRD), or short memory or short-range dependence (SRD). A stationary Gaussian process with covariance $C(t)$ is said to be LRD if for some finite $t_*\geqslant0$,

$$\int_{t_*}^{\infty}|C(t)|\,\mathrm{d}\tau=\infty,\tag{3.21}$$

otherwise it is SRD. From the results (3.19) and (3.20), we find that the covariance of $X_{\alpha,\gamma}(t)$ behaves asymptotically as $C_{\alpha,\gamma}(t)\sim t^{-(2\alpha+1)}$ if $\alpha\in(0,1)$ and as $C_{\alpha,\gamma}(t)\sim\mathrm{e}^{-\lambda t}t^{\gamma-1}$ if $\alpha=1$, for $t\to\infty$. This shows that the corresponding integral (3.21) is convergent and therefore the fractional oscillator process $X_{\alpha,\gamma}(t)$ has SRD. We remark that another way to characterize a short memory process is that the spectral density of the process is continuous at the origin. For the process $X_{\alpha,\gamma}(t)$, $S_{\alpha,\gamma}(0)$ is just $1/(2\pi\lambda^{2\gamma})$.

It is interesting to note that when $\alpha\in(0,1)$ the asymptotic order of the covariance does not depend on the parameter γ. Therefore, the parameter α characterizes the asymptotic order

of the covariance $C_{\alpha,\gamma}(t)$ as $t \to \infty$. Combining with the remark given earlier, one notes that it is possible to separately characterize the fractal dimension and short-range dependence of $X_{\alpha,\gamma}(t)$ with two different indices.

4. $X_{\alpha,\gamma}(t)$ as a velocity process

Diffusion is one of the basic non-equilibrium phenomena. Normal diffusion $Y(t)$ is characterized by a mean-square displacement that is asymptotically linear in time, i.e. $\langle Y(t)^2 \rangle = 2Dt$, where D is the diffusion constant [57]. However, experiments show that there are complex processes whose mean-square displacements behave like $\langle Y(t)^2 \rangle \sim t^\kappa, 0 < \kappa < 2$. These processes are called anomalous diffusions and they have been observed in various physical processes (see the references in [3–5, 58]).

In this section, we assume that the fractional oscillator process $X_{\alpha,\gamma}(t)$ can be regarded as a velocity process, and we would like to consider some of the consequences: in particular, the possibility of the extension of fluctuation–dissipation theorem [1] and the possibility of using $X_{\alpha,\gamma}(t)$ to model anomalous diffusion.

One of the important theorems in statistical mechanics is the fluctuation–dissipation theorem [1] which relates the coefficient of the covariance of the external random force in the Langevin equation to the frictional coefficient. It would be interesting to see whether there exists some kind of fluctuation–dissipation relation for the fractional process $X_{\alpha,\gamma}(t)$. For this purpose we re-express the fractional Langevin equation (2.13) as

$$\left(\mathbf{D}_t^{2\alpha} + \lambda^{2\alpha}\right)^{\frac{\gamma}{2}} X_{\alpha,\gamma}(t) = \eta(t), \tag{4.1}$$

with the covariance of white noise $\eta(t)$ given by $\langle \eta(t)\eta(s) \rangle = 2B\delta(t - s)$, where B is a constant coefficient. We first consider the position (displacement) process $Y_{\alpha,\gamma}(t)$ whose ordinary derivative gives $X_{\alpha,\gamma}(t)$, i.e.

$$X_{\alpha,\gamma}(t) = \frac{dY_{\alpha,\gamma}(t)}{dt}. \tag{4.2}$$

More generally, since the evolution of the velocity $X_{\alpha,\gamma}(t)$ is described by fractional dynamics (4.1), we will consider the velocity linked to the displacement by the following relation:

$$X_{\alpha,\gamma}(t) = {}_0D_t^\chi Y_{\alpha,\gamma}(t), \qquad \tfrac{1}{2} < \chi < \tfrac{3}{2}, \tag{4.3}$$

where ${}_0D_t^\chi$ denotes the Riemann–Liouville fractional derivative of order χ. In other words, the velocity is the fractional derivative of order χ of the position. Here we would like to remark that the representation of the process $X_{\alpha,\gamma}(t)$ by (2.12) has a drawback of lacking satisfactory causal interpretation. Nevertheless, we partially rectify this problem by using the Riemann–Liouville fractional derivative in definition (4.3) of the position process. From another point of view, the discussions below only involve the long-time asymptotic behavior of the processes $X_{\alpha,\gamma}(t)$ and $Y_{\alpha,\gamma}(t)$, the results are valid for any process having asymptotically the same behavior as $X_{\alpha,\gamma}(t)$. One can regard the process $X_{\alpha,\gamma}(t)$ below as a process whose long-time equilibrium limit is described by (4.1). Alternatively, one can also substitute the process $X_{\alpha,\gamma}(t)$ by the process $\mathsf{X}_{\alpha,\gamma}(t)$ (2.6) which has another representation [30]

$$\mathsf{X}_{\alpha,\gamma}(t) = \int_{-\infty}^t K_{\alpha,\gamma}(t - u)\eta(u)\,du$$

that admits causal interpretation. Here

$$K_{\alpha,\beta}(t) = \frac{\mathrm{Im}}{\pi} \int_0^\infty \frac{e^{-ut}}{(u^\alpha\, e^{-i\pi\alpha} + \lambda)^\gamma}\,du$$

is the memory kernel of $\mathsf{X}_{\alpha,\gamma}(t)$.

Returning to our discussion about the fluctuation–dissipation relation, the assumption of the thermalization of the fractional velocity process based on the fractional generalization of the classical principle of equipartition of energy [1] is

$$\langle [X_{\alpha,\gamma}(t)]^2 \rangle = \varsigma (kT)^\theta, \tag{4.4}$$

where we have assumed the particle under consideration has unit mass, k is the Boltzmann constant, T denotes temperature and ς is a constant which carries suitable physical dimensions to render (4.4) dimensionally consistent.

Under the condition $\alpha\gamma > 1/2$, the process $X_{\alpha,\gamma}(t)$ has a finite variance given by (3.4):

$$C_{\alpha,\gamma}(0) = \frac{B}{\pi\alpha} \frac{\Gamma\left(\frac{1}{2\alpha}\right)\Gamma\left(\gamma - \frac{1}{2\alpha}\right)}{\Gamma(\gamma)} \lambda^{1-2\alpha\gamma}. \tag{4.5}$$

From (4.4) and (4.5) one obtains

$$B = \frac{\pi\alpha\Gamma(\gamma)}{\Gamma\left(\frac{1}{2\alpha}\right)\Gamma\left(\gamma - \frac{1}{2\alpha}\right)} \varsigma \lambda^{2\alpha\gamma-1}(kT)^\theta = n(\alpha,\gamma)\varsigma\lambda^{2\alpha\gamma-1}(kT)^\theta, \tag{4.6}$$

with

$$n(\alpha,\gamma) = \frac{\pi\alpha\Gamma(\gamma)}{\Gamma\left(\frac{1}{2\alpha}\right)\Gamma\left(\gamma - \frac{1}{2\alpha}\right)}.$$

Equation (4.6) can be regarded as the generalized fluctuation–dissipation relation for the fractional velocity process. When $\alpha = 1$, $\gamma = 1$ and $\theta = 1$, $\varsigma = 1$ and (4.6) reduces to the fluctuation–dissipation relation for the ordinary Ornstein–Uhlenbeck process $X_{1,1}(t)$:

$$B = \lambda kT. \tag{4.7}$$

We would like to remark that the noise $\eta(t)$ we consider here is an external noise. If the noise is internal, some modifications are required in order to satisfy the fluctuation–dissipation relation. We plan to extend our work to internal noise in a future work.

Now consider the displacement process $Y_{\alpha,\gamma}(t)$. We can show that for a short-range dependent process such as $X_{\alpha,\gamma}(t)$, the leading term for the large time behavior of the variance of its mean-square displacement does not depend on the covariance of $X_{\alpha,\gamma}(t)$. For illustration, let us first consider the simple case with the position process $Y_{\alpha,\gamma}(t)$ linked to the velocity process by ordinary derivative (4.2). If we assume that $Y_{\alpha,\gamma}(0) = 0$, then the variance of the position process $Y_{\alpha,\gamma}(t)$ is given by

$$\langle [Y_{\alpha,\gamma}(t)]^2 \rangle = \int_0^t \int_0^t C_{\alpha,\gamma}(|s_1 - s_2|) \, ds_2 \, ds_1. \tag{4.8}$$

By some calculus, we have

$$\langle [Y_{\alpha,\gamma}(t)]^2 \rangle = \int_0^t \int_0^{s_1} C_{\alpha,\gamma}(s_1 - s_2) \, ds_2 \, ds_1 + \int_0^t \int_{s_1}^t C_{\alpha,\gamma}(s_2 - s_1) \, ds_1 \, ds_2$$

$$= \int_0^t \int_0^s C_{\alpha,\gamma}(\tau) \, d\tau \, ds + \int_0^t \int_0^{s_2} C_{\alpha,\gamma}(s_2 - s_1) \, ds_1 \, ds_2$$

$$= 2 \int_0^t \int_0^s C_{\alpha,\gamma}(\tau) \, d\tau \, ds = 2 \int_0^t (t - \tau) C_{\alpha,\gamma}(\tau) \, d\tau. \tag{4.9}$$

Since $C_{\alpha,\gamma}(\tau) \sim \tau^{-2\alpha-1}$ as $\tau \to \infty$, the integral $\int_0^\infty C_{\alpha,\gamma}(\tau) \, d\tau$ is convergent and we find that in the long-time ($t \gg 1$) limit,

$$\langle [Y_{\alpha,\gamma}(t)]^2 \rangle \sim 2 \left[\int_0^\infty C_{\alpha,\gamma}(\tau) \, d\tau \right] t. \tag{4.10}$$

This shows that $Y_{\alpha,\gamma}(t)$ is an ordinary diffusion with diffusion constant

$$D = \int_0^\infty C_{\alpha,\gamma}(\tau)\,\mathrm{d}\tau = \pi S_{\alpha,\gamma}(0) = B\lambda^{-2\alpha\gamma}. \qquad (4.11)$$

Using (4.6), one gets

$$D = \frac{\pi\alpha\Gamma(\gamma)\varsigma}{\Gamma\left(\frac{1}{2\alpha}\right)\Gamma\left(\gamma - \frac{1}{2\alpha}\right)}\frac{(kT)^\theta}{\lambda},$$

which reduces to the well-known Einstein relation

$$D = \frac{kT}{\lambda}$$

for $\alpha = \gamma = \theta = 1$. This simplified example shows that the long-time behavior of $C_{\alpha,\gamma}(\tau)$ does not show up in the leading term of the long-time asymptotic expression of the variance $\langle[Y_{\alpha,\gamma}(t)]^2\rangle$. Its effect only appears in the second leading term (see appendix A). This is due to the fact that $X_{\alpha,\gamma}(t)$ is a short-range process with its covariance $C(\tau) \sim \tau^{-2\alpha-1}, \alpha > 0$ for $\tau \to \infty$. We remark that (4.10) is consistent with the result obtained for the case with $\alpha > 0, \gamma = 1$ if the usual velocity–displacement relation (4.2) is used [3].

Now we consider the fractional case with the velocity linked to the displacement by relation (4.3). If we further assume that $\left.{}_0D_t^{\chi-j}Y_{\alpha,\gamma}(t)\right|_{t=0} = 0$ for $j = 1$ if $\chi \leqslant 1$ and $j = 1, 2$ if $\chi > 1$, then the position process is given by

$$Y_{\alpha,\gamma}(t) = {}_0I_t^\chi X_{\alpha,\gamma}(t) = \frac{1}{\Gamma(\chi)}\int_0^t (t-u)^{\chi-1}X_{\alpha,\gamma}(u)\,\mathrm{d}u. \qquad (4.12)$$

One can show that (see appendix A)

$$\langle[Y_{\alpha,\gamma}(t)]^2\rangle = 2B\lambda^{-2\alpha\gamma}\left[\frac{t^{2\chi-1}}{(2\chi-1)\Gamma(\chi)^2}\right] + O(t^{\max\{0,2\chi-2,2\chi-2\alpha-1\}}\log t). \qquad (4.13)$$

The term in the bracket on the right-hand side of (4.13) is just the variance of the Riemann–Liouville fractional Brownian motion indexed by $\chi - 1/2$ [13]. The behavior $\langle[Y_{\alpha,\gamma}(t)]^2\rangle \sim t^{2\chi-1}$ shows that $Y_{\alpha,\gamma}(t)$ is a super-diffusion for $\chi > 1$ and a sub-diffusion for $\chi < 1$. Thus $Y_{\alpha,\gamma}(t)$ provides a different model for anomalous diffusion [3–5]. Note that the anomalous order χ is independent of the fractional order α and γ of the field $X_{\alpha,\gamma}(t)$. However, one can show that the regularity of the path $Y_{\alpha,\gamma}(t)$ is determined by the parameter $\alpha\gamma + \chi$.

Since the total fractional derivative order of $X_{\alpha,\gamma}(t)$ in (4.1) is $\alpha\gamma$, we may consider setting $\chi = \alpha\gamma$ in (4.3). Then substituting B from (4.6), we get

$$\langle[Y_{\alpha,\gamma}(t)]^2\rangle \sim \frac{2\pi\alpha\Gamma(\gamma)}{\Gamma\left(\frac{1}{2\alpha}\right)\Gamma\left(\gamma - \frac{1}{2\alpha}\right)}\varsigma\frac{(kT)^\theta}{\lambda}\left[\frac{t^{2\alpha\gamma-1}}{(2\alpha\gamma-1)\Gamma(\alpha\gamma)^2}\right]$$

$$\sim N(\alpha,\gamma)\varsigma\frac{(kT)^\theta}{\lambda}t^{2\alpha\gamma-1}, \qquad (4.14)$$

where $N(\alpha,\gamma)$ is a constant term depending on α and γ.

Finally we consider the Fokker–Planck equation of the process $Y_{\alpha,\gamma}(t)$. Let $P(y,t)$ denote the probability of finding $Y_{\alpha,\gamma}(t)$ at the point y at time t. Then since $Y_{\alpha,\gamma}(t)$ is a Gaussian process, we have

$$P(y,t) = \frac{1}{\sqrt{2\pi}\sigma_Y(t)}\exp\left(-\frac{y^2}{2\sigma_Y^2(t)}\right), \qquad (4.15)$$

where

$$\sigma_Y^2(t) = \langle Y(t)^2\rangle.$$

Using (4.15), one can easily check that $P(y, t)$ satisfies the effective Fokker–Planck equation

$$\frac{\partial}{\partial t} P(y, t) = D(t) \frac{\partial^2}{\partial x^2} P(y, t), \tag{4.16}$$

with the diffusion coefficient $D(t)$ given by [3, 34]

$$D(t) = \frac{1}{2} \frac{\partial}{\partial t} \langle [Y_{\alpha, \gamma}(t)]^2 \rangle.$$

When $\chi = \alpha\gamma$, the equilibrium state (i.e., $t \to \infty$) of the mean-square displacement $\langle [Y_{\alpha, \gamma}(t)]^2 \rangle$ is given by (4.14). Therefore

$$D(t) \sim \frac{1}{2} N(\alpha, \gamma) \varsigma \frac{(kT)^\theta}{\lambda} (2\alpha\gamma - 1) t^{2\alpha\gamma - 2},$$

which up to the constant $N(\alpha, \gamma)$, is the effective diffusion coefficient for fractional Brownian motion if we let $\alpha\gamma - 1/2 = H$, the Hurst index [59–61]. Our result differs from that of [59–61] which has kT instead of the $\varsigma(kT)^\theta$ given above. This is due to our use of the fractional generalization of equipartition principle (4.4).

Here we would also like to remark that since $Y_{\alpha, \gamma}(t)$ is a non-Markovian process, the Fokker–Planck equation (4.16) cannot fully define the process [62]. Nevertheless, since $Y_{\alpha, \gamma}(t)$ is a Gaussian process, it is explicitly determined by the mean $\langle Y_{\alpha, \gamma}(t) \rangle = 0$ and the covariance function

$$\langle Y_{\alpha, \gamma}(t) Y_{\alpha, \gamma}(s) \rangle = \frac{1}{\Gamma(\chi)^2} \int_0^t \int_0^s (t - u)^{\chi - 1} (s - v)^{\chi - 1} C_{\alpha, \gamma}(u - v) \, \mathrm{d}v \, \mathrm{d}u,$$

where $C_{\alpha, \gamma}(t)$ is given by (2.11).

From the discussion above, we see that the long-time dependence of the covariance of $X_{\alpha, \gamma}(t)$ does not enter in the leading term of the variance of $Y_{\alpha, \gamma}(t)$. It only appears as a second leading term. Instead, the property of the leading term depends only on the differential relationship between $X_{\alpha, \gamma}(t)$ and $Y_{\alpha, \gamma}(t)$. We would like to emphasize again that, in order to derive the anomalous order of the displacement process $Y_{\alpha, \gamma}(t)$ as a diffusion, we only use the fact that the velocity process is a short-range process and it is the derivative of order χ of the velocity process $X_{\alpha, \gamma}(t)$. The result in this section is valid if we consider any short-range process as the velocity process.

In the following, we return to the case where $\langle \eta(t) \eta(t') \rangle = \delta(t - t')$.

5. Casimir energy associated with $X_{\alpha, \gamma}(t)$ at finite temperature

The ordinary oscillator process can be regarded as a one-dimensional Euclidean scalar massive field as its spectral density $(\omega^2 + \lambda^2)^{-1}$ is just the Euclidean propagator for a scalar field with mass λ. By analogy, one can consider the fractional oscillator process $X_{\alpha, \gamma}(t)$ as a fractional Euclidean scalar massive field in one dimension with propagator $(|\omega|^{2\alpha} + \lambda^2)^{-\gamma}$. From this viewpoint, it will be interesting to find a convenient way to quantize the corresponding fractional quantum field $\phi_{\alpha, \gamma}(t)$. This can be achieved by using the stochastic quantization of Parisi and Wu [48]. According to this quantization scheme, an additional auxiliary time τ is introduced and the Euclidean quantum field $\phi_{\alpha, \gamma}(t; \tau)$ is assumed to evolve in this auxiliary time according to a stochastic differential equation of Langevin type with external white noise. The large equal-τ equilibrium limit of the covariance function of the solution to this Langevin equation gives the one-dimensional two-point Schwinger function of the Euclidean field $\phi_{\alpha, \gamma}(t)$.

The nonlocal Euclidean action of the massive scalar field which satisfies the (one-dimensional) fractional Klein–Gordon equation

$$[(-\Delta)^\alpha + \lambda^2]^\gamma \phi_{\alpha,\gamma}(t) = 0$$

is given by

$$S[\phi_{\alpha,\gamma}] = \frac{1}{2} \int_{\mathbb{R}} \phi_{\alpha,\gamma}(t)[(-\Delta)^\alpha + \lambda^2]^\gamma \phi_{\alpha,\gamma}(t) \, dt. \tag{5.1}$$

The Parisi–Wu quantization procedure requires $\phi_{\alpha,\gamma}(t; \tau)$ to satisfy the following stochastic differential equation:

$$\frac{\partial \phi_{\alpha,\gamma}(t; \tau)}{\partial \tau} = -\left.\frac{\delta S[\phi_{\alpha,\gamma}]}{\delta \phi_{\alpha,\gamma}}\right|_{\phi_{\alpha,\gamma}=\phi_{\alpha,\gamma}(t;\tau)} + \eta(t; \tau), \tag{5.2}$$

where $\eta(t, \tau)$ is the external white noise defined by

$$\langle \eta(t; \tau) \rangle = 0, \qquad \langle \eta(t; \tau)\eta(t'; \tau') \rangle = 2\delta(t - t')\delta(\tau - \tau').$$

Equations (5.1) and (5.2) give

$$\frac{\partial \phi_{\alpha,\gamma}(t; \tau)}{\partial \tau} = -[(-\Delta)^\alpha + \lambda^2]^\gamma \phi_{\alpha,\gamma}(t) + \eta(t; \tau). \tag{5.3}$$

The solution of (5.3) subjected to the initial condition $\phi_{\alpha,\gamma}(t; 0) = 0$ is

$$\phi_{\alpha,\gamma}(t; \tau) = \int_{\mathbb{R}} \int_0^\tau G(t - t'; \tau - \tau')\eta(t'; \tau') \, dt' \, d\tau', \tag{5.4}$$

where $G(t; \tau)$ is the retarded Green function given by

$$G(t; \tau) = \frac{\theta(\tau)}{2\pi} \int_{\mathbb{R}} \exp[-(|\omega|^{2\alpha} + \lambda^2)^\gamma \tau] \, e^{i\omega t} \, d\omega.$$

The large equal-τ limit ($\tau = \tau' \to \infty$) of the covariance function gives

$$\lim_{\tau_1=\tau_2\to\infty} \langle \phi_{\alpha,\gamma}(t_1, \tau_1)\phi_{\alpha,\gamma}(t_2, \tau_2) \rangle = 2 \lim_{\tau\to\infty} \int_{\mathbb{R}} \int_0^\tau G(t_1 - t'; \tau - \tau')G(t_2 - t'; \tau - \tau') \, d\tau' \, dt'$$

$$= \frac{1}{\pi} \int_{\mathbb{R}} \int_0^\infty \exp[-2(|\omega|^{2\alpha} + \lambda^2)^\gamma \tau'] \, e^{i\omega(t_1-t_2)} \, d\tau' \, d\omega$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}} \frac{e^{i\omega(t_1-t_2)}}{(|\omega|^{2\alpha} + \lambda^2)^\gamma} \, d\omega, \tag{5.5}$$

which is just the Euclidean two-point function of the fractional field $\phi_{\alpha,\gamma}(t)$.

We next consider the Parisi–Wu quantization method at positive temperature $T = \beta^{-1}$. We follow the Matsubara imaginary time formalism of finite temperature field theory by requiring $\phi_{\alpha,\gamma}^T(t; \tau)$ to be periodic in the Euclidean time t with period β. That is,

$$\phi_{\alpha,\gamma}^T(t + \beta; \tau) = \phi_{\alpha,\gamma}^T(t; \tau). \tag{5.6}$$

In addition, the white noise $\eta^T(t; \tau)$ is assumed to satisfy the periodic condition in t, i.e. $\eta^T(t + \beta; \tau) = \eta^T(t; \tau)$, such that

$$\langle \eta^T(t; \tau) \rangle = 0, \qquad \langle \eta^T(t; \tau)\eta^T(t'; \tau') \rangle = \frac{2}{\beta} \sum_{n=-\infty}^{\infty} \exp[i\omega_n(t - t')]\delta(\tau - \tau'), \tag{5.7}$$

with $\omega_n = 2\pi n/\beta$. The fractional operator $[(-\Delta)^{2\alpha} + \lambda^2]^\gamma$ acting on $\phi_{\alpha,\gamma}^T(t; \tau)$ is defined via Fourier series expansion (with respect to t) of $\phi_{\alpha,\gamma}^T(t; \tau)$, i.e.,

$$[(-\Delta)^{2\alpha} + \lambda^2]^\gamma \phi_{\alpha,\gamma}^T(t; \tau) = \frac{1}{\beta} \sum_{n=-\infty}^{n=\infty} \left[\int_0^\beta \phi_{\alpha,\gamma}(t'; \tau) \, e^{-i\omega_n t'} \, dt' \right] (|\omega_n|^{2\alpha} + \lambda^2)^\gamma \, e^{i\omega_n t}.$$

The retarded Green function for the Langevin equation (5.2) satisfying the periodic conditions is given by

$$G^T(t; \tau) = \frac{\theta(\tau)}{\beta} \sum_{n=-\infty}^{\infty} \exp[-(|\omega_n|^{2\alpha} + \lambda^2)^\gamma \tau]\, e^{i\omega_n t}, \tag{5.8}$$

so that the solution with initial condition $\phi_{\alpha;\gamma}^T(t; 0) = 0$ is

$$\phi_{\alpha,\gamma}^T(t; \tau) = \int_0^\beta \int_0^\tau G^T(t - t'; \tau - \tau')\eta^T(t'; \tau')\, d\tau'\, dt', \tag{5.9}$$

with covariance given by

$$\langle \phi_{\alpha,\gamma}^T(t_1, \tau_1)\phi_{\alpha,\gamma}^T(t_2, \tau_2) \rangle = \frac{2}{\beta} \sum_{n=-\infty}^{\infty} e^{i\omega_n(t_1-t_2)} \int_0^{\tau_1 \wedge \tau_2} \exp[-(|\omega_n|^{2\alpha} + \lambda^2)^\gamma (\tau_1 - \tau')]$$

$$\times \exp[-(|\omega_n|^{2\alpha} + \lambda^2)^\gamma (\tau_2 - \tau')]\, d\tau'. \tag{5.10}$$

At the large equal-τ limit, the covariance becomes

$$\lim_{\tau_1=\tau_2\to\infty} \langle \phi_{\alpha,\gamma}^T(t_1, \tau_1)\phi_{\alpha,\gamma}^T(t_2, \tau_2) \rangle = \frac{1}{\beta} \sum_{n=-\infty}^{\infty} \frac{e^{i\omega_n(t_1-t_2)}}{(|\omega_n|^{2\alpha} + \lambda^2)^\gamma}, \tag{5.11}$$

which is the thermal two-point function for the Euclidean fractional Klein–Gordon field. When $\alpha = \gamma = 1$, this reduces to the ordinary two-point function of one-dimensional Euclidean scalar field at finite temperature [63]. We would also like to mention that when $\beta \to \infty$, the limit of the thermal two-point function (5.11) is the two-point function (5.5).

We remark that we can also consider the solution $\phi_{\alpha,\gamma}^\infty(t; \tau)$ to (5.2) satisfying the initial condition $\phi_{\alpha,\gamma}^\infty(t; -\infty) = 0$, i.e., instead of having the field evolving from $\tau = 0$, we require it to evolve from $\tau = -\infty$. The solution $\phi_{\alpha,\gamma}^\infty(t; \tau)$ is then given by

$$\phi_{\alpha,\gamma}^\infty(t; \tau) = \int_\mathbb{R} \int_{-\infty}^\tau G(t - t'; \tau - \tau')\eta(t'; \tau')\, dt'\, d\tau'. \tag{5.12}$$

It can be shown that $\phi_{\alpha,\gamma}^\infty(t; \tau)$ is a stationary field and in the large-τ limit, the field $\phi_{\alpha,\gamma}(t; \tau)$ approaches the field $\phi_{\alpha,\gamma}^\infty(t; \tau)$, i.e.,

$$\lim_{\tau\to\infty} \phi_{\alpha,\gamma}(t; \tau) = \phi_{\alpha,\gamma}^\infty(t; \tau).$$

The equal-τ covariance of $\phi_{\alpha,\gamma}^\infty(t; \tau)$ is independent of τ and is precisely the propagator (5.5). The same statement applies to $\phi_{\alpha,\gamma}^{T,\infty}(t; \tau)$ which is the solution to the periodic version of (5.2) with boundary condition $\phi_{\alpha,\gamma}^{T,\infty}(t; -\infty) = 0$.

We would also like to remark that just as the stochastic process $X_{\alpha,\beta}(t)$ is related to the field $\phi_{\alpha,\beta}(t)$ in the sense that the covariance function of $X_{\alpha,\beta}(t)$ coincides with the propagator of $\phi_{\alpha,\beta}(t)$, we can define a periodic stochastic process $X_{\alpha,\beta}^T(t)$ whose covariance function is the propagator of the periodic field $\phi_{\alpha,\gamma}^T(t)$ (5.11). In fact, consider the solution of the fractional stochastic differential equation

$$[(-\Delta)^\alpha + \lambda^2]^{\frac{\gamma}{2}} X_{\alpha,\gamma}^T(t) = \eta^T(t),$$

where $\eta^T(t)$ is the periodic white noise with period β and

$$\langle \eta^T(t) \rangle = 0, \qquad \langle \eta^T(t)\eta^T(t') \rangle = \frac{1}{\beta} \sum_{n=-\infty}^{\infty} e^{i\omega_n(t-t')}.$$

Using Fourier series, it is easy to check that the solution is given by

$$X_{\alpha,\beta}^T(t) = \frac{1}{\beta} \sum_{n=-\infty}^{\infty} \int_0^\beta \frac{e^{i\omega_n(t-t')}}{(|\omega_n|^{2\alpha} + \lambda^2)^{\frac{\gamma}{2}}} \eta^T(t') \, dt',$$

and the covariance function is

$$\langle X_{\alpha,\gamma}^T(t) X_{\alpha,\gamma}^T(t') \rangle = \frac{1}{\beta} \sum_{n=-\infty}^{\infty} \frac{e^{i\omega_n(t_1-t_2)}}{(|\omega_n|^{2\alpha} + \lambda^2)^\gamma},$$

which coincides with the thermal two-point function (5.11).

The fact that the Schwinger two-point function of the quantum field $\phi_{\alpha,\gamma}(t)$ is the same as the covariance function of the stochastic process $X_{\alpha,\gamma}(t)$ allows us to apply the result in section 3 to conclude that

$$\langle \phi_{\alpha,\gamma}(t) \phi_{\alpha,\gamma}(t') \rangle \sim \frac{1}{|t-t'|^{1-2\alpha\gamma}}$$

when $2\alpha\gamma < 1$. Namely, the field $\phi_{\alpha,\gamma}(t)$ has scaling dimension $\frac{1}{2} - \alpha\gamma$ if $\alpha\gamma < 1/2$. This should be compared to the fact that the stochastic process has Hölder exponent $\alpha\gamma - 1/2$ when $\alpha\gamma > 1/2$. On the other hand, the relation between the quantum field $\phi_{\alpha,\gamma}(t)$ and the stochastic process $X_{\alpha,\gamma}(t)$ does not restrict to the coincidence of the propagator of the former and the covariance of the later. One can actually show that the physically interesting n-point correlation function of the quantum field $\phi_{\alpha,\gamma}(t)$—$\langle \phi_{\alpha,\gamma}(t_1) \cdots \phi_{\alpha,\gamma}(t_n) \rangle$, is also given by the multi-covariance function $\langle X_{\alpha,\gamma}(t_1) \cdots X_{\alpha,\gamma}(t_n) \rangle$.

In statistical mechanics, the partition function is an important quantity that encodes the statistical properties of a system in thermodynamic equilibrium. Just as the partition function of a simple harmonic oscillator coincides with the partition function of the massive scalar field in one dimension, we can envisage such a relation to hold for the fractional oscillator $X_{\alpha,\gamma}(t)$ and the quantum field $\phi_{\alpha,\gamma}(t)$. Therefore we proceed to calculate the partition function for the fractional Euclidean field $\phi_{\alpha,\gamma}^T(t)$ in one dimension at finite temperature T, and its associated Casimir free energy. For this purpose, we employ the technique of zeta function regularization [64–67]. Due to the fractional character of the scalar field under consideration, the derivation of the Casimir free energy is more complicated as compared to the ordinary scalar field.

By definition, the Casimir free energy $\mathcal{F}$ of the fractional Klein–Gordon field $\phi_{\alpha,\gamma}^T(t)$ which is kept at thermal equilibrium with temperature $T = \beta^{-1}$ is given by

$$\mathcal{F} = -\frac{1}{\beta} \log \mathcal{Z},$$

where $\mathcal{Z}$ is the partition function defined by

$$\mathcal{Z} = \int \mathcal{D}\phi^T \exp\left(-\frac{1}{2} \int_0^\beta \phi^T(t)[(-\Delta)^\alpha + m^2]^\gamma \phi^T(t) \, dt \right).$$

By using zeta regularization techniques, we find that

$$\mathcal{F} = -\frac{1}{2\beta}(\zeta'(0) - \zeta(0) \log \mu^2), \tag{5.13}$$

where μ is a normalization constant and $\zeta(s)$ is the zeta function

$$\zeta(s) = m^{-2\gamma s} + 2 \sum_{n=1}^{\infty} \{[an]^{2\alpha} + m^2\}^{-\gamma s}, \tag{5.14}$$

with $a = \frac{2\pi}{\beta}$. The series in (5.14) is divergent when $s \leqslant 1/(2\alpha\gamma)$. Therefore we need to find an analytic continuation of $\zeta(s)$ to a neighborhood of $s = 0$. The computation is quite involved and we leave it to appendix B. The result is

$$
\mathcal{F} = \frac{1}{2\beta}\left\{ \omega_{\alpha,\Lambda}(-1)^{\frac{1}{2\alpha}}\frac{\beta}{\pi}m^{\frac{1}{\alpha}}\left[\log\mu^2 - \gamma\left(\psi\left(\frac{1}{2\alpha}+1\right) - \psi(1) - \log m^2 \right) \right] \right.
$$
$$
\left. + \gamma(1-\omega_{\alpha,\Lambda})\frac{\beta}{\sin\frac{\pi}{2\alpha}}m^{\frac{1}{\alpha}} - \gamma\int_0^\infty t^{-1}K(t)\,\mathrm{e}^{-tm^2}\,\mathrm{d}t \right\}.
$$

Here

$$
K(t) = 2\sum_{n=1}^\infty \mathrm{e}^{-t[an]^{2\alpha}} - \frac{1}{\alpha}\Gamma\left(\frac{1}{2\alpha}\right)t^{-\frac{1}{2\alpha}}a^{-1} + 1,
$$

and

$$
\omega_{\alpha,\Lambda} = \begin{cases} 1, & \text{if}\quad \alpha\in\Lambda = \left\{ \frac{1}{2u}: u\in\mathbb{N}\right\} \\ 0, & \text{if}\qquad\quad \alpha\notin\Lambda. \end{cases}
$$

When $\beta\to\infty$, we have the asymptotic behavior (see appendix B):

$$
\mathcal{F} \sim \omega_{\alpha,\Lambda}\frac{(-1)^{\frac{1}{2\alpha}}}{2\pi}m^{\frac{1}{\alpha}}\left[\log\mu^2 - \gamma\left(\psi\left(\frac{1}{2\alpha}+1\right) - \psi(1) - \log m^2 \right) \right]
$$
$$
+ \gamma(1-\omega_{\alpha,\Lambda})\frac{m^{\frac{1}{\alpha}}}{2\sin\frac{\pi}{2\alpha}} - \gamma\sum_{k=1}^\infty \frac{(-1)^k}{k}m^{-2k}(2\pi)^{2\alpha k}\zeta_R(-2\alpha k)\beta^{-2\alpha k-1}. \quad (5.15)
$$

In the special case $\alpha = 1$, since $\zeta(-2k) = 0$ for all $k\in\mathbb{N}$, (5.15) gives us

$$
\mathcal{F} \sim \frac{\gamma m}{2}. \tag{5.16}
$$

In fact, when $\alpha = 1$, we have the explicit formula (see appendix B):

$$
\mathcal{F} = \frac{\gamma m}{2} + \frac{\gamma}{\beta}\log(1-\mathrm{e}^{-\beta m}) = \frac{\gamma}{\beta}\log\left[2\sinh\frac{\beta m}{2}\right]. \tag{5.17}
$$

It is easy to see that the leading term when $\beta\gg 1$ agrees with (5.16) and the remainder terms decay exponentially.

From (5.15), we note that at low temperature $T\ll 1$ ($\beta\gg 1$), the leading order term of the free energy $\mathcal{F}$ is of order T^0. When α is not the reciprocal of an even number, then the leading order is

$$
\mathcal{F} \sim \frac{\gamma m^{\frac{1}{\alpha}}}{2\sin\frac{\pi}{2\alpha}} + O(T^{1+2\alpha}), \tag{5.18}
$$

whose sign depends on α. There is a dependence of $\mathcal{F}$ on the normalization constant μ when α is the reciprocal of an even number. We shall renormalize the free energy $\mathcal{F}$ to get rid of this dependence later.

Formula (5.15) is not suitable for studying the high temperature behavior of the Casimir free energy $\mathcal{F}$. In appendix B, we derive the following alternative expression for the free energy $\mathcal{F}$:

$$
\mathcal{F} = \frac{1}{2\beta}\left\{ \gamma\log m^2 + 2\alpha\gamma\log\beta - 2\gamma\sum_{\substack{l\in\mathbb{N} \\ l\neq\frac{1}{2\alpha}}} \frac{(-1)^l}{l}m^{2l}\left(\frac{\beta}{2\pi}\right)^{2\alpha l}\zeta_R(2\alpha l) + \omega_{\alpha,\Lambda}(-1)^{\frac{1}{2\alpha}}\frac{\beta}{\pi}m^{\frac{1}{\alpha}} \right.
$$
$$
\left. \times\left[\log\mu^2 + \alpha\gamma\left(\log\left(\frac{2\pi}{\beta}\right)^2 + 2\psi(1)\right) - \gamma\left[\psi\left(\frac{1}{2\alpha}\right) - \psi(1)\right]\right]\right\}, \quad (5.19)
$$

which is valid when $\beta < \frac{2\pi}{m}$. In particular, in the high temperature limit $T \gg 1$ ($\beta \ll 1$),

$$\mathcal{F} \sim -\alpha\gamma T \log T + \frac{\gamma T}{2} \log m^2 + O(1), \tag{5.20}$$

the Casimir free energy is negative and the leading term $-\alpha\gamma T \log T$ depends linearly on α and γ. When $\alpha = 1$, using

$$\zeta_R(2l) = (-1)^{l+1} \frac{(2\pi)^{2l}}{2(2l)!} B_{2l},$$

where $B_{2l}, l \geqslant 1$ are the Bernoulli numbers defined by

$$\frac{1}{e^x - 1} = \frac{1}{x} - \frac{1}{2} + \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!} x^{2n-1},$$

and

$$\log \frac{1 - e^{-x}}{x} = \int_0^x \left\{ \frac{1}{e^u - 1} - \frac{1}{u} \right\} du = -\frac{x}{2} + \frac{1}{2} \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!} \frac{x^{2n}}{n},$$

we find that (5.19) gives

$$\mathcal{F} = \frac{1}{2\beta} \left\{ \gamma \log m^2 + 2\gamma \log \beta + \gamma \sum_{l=1}^{\infty} \frac{B_{2l}}{(2l)!} \frac{(\beta m)^{2l}}{l} \right\}$$

$$= \frac{\gamma}{2\beta} \left\{ 2\log[\beta m] + 2\log\left[\frac{1 - e^{-\beta m}}{\beta m} \right] + \beta m \right\} = \frac{\gamma}{\beta} \log\left[2\sinh\frac{\beta m}{2} \right],$$

agreeing with (5.17).

One notes that when α is the reciprocal of an even number, the free energy depends on the normalization constant μ. In order to remove this dependence, we need to renormalize the free energy by adding a counterterm $\mathcal{F}_c$ to the free energy so that the renormalized free energy $\mathcal{F}_{\text{ren}}$ is

$$\mathcal{F}_{\text{ren}} = \mathcal{F} + \mathcal{F}_c.$$

A reasonable way to determine the counterterm $\mathcal{F}_c$ is to require that in the limit $\beta \to \infty$ and $m \to 0$, $\mathcal{F}_{\text{ren}} \to 0$. equation (5.15) gives us immediately

$$\mathcal{F}_c = -\omega_{\alpha,\Lambda} \frac{(-1)^{\frac{1}{2\alpha}}}{2\pi} m^{\frac{1}{\alpha}} (\log \mu^2 + \gamma \log m^2).$$
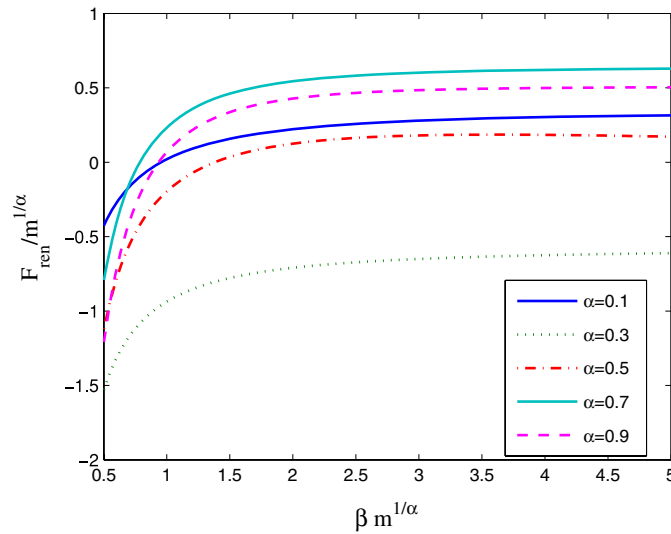
Note that adding the counterterm $\mathcal{F}_c$ to the free energy is equivalent to setting $\mu = m^{-\gamma}$ when α is the reciprocal of an even integer. Therefore, from (5.15) and (5.19), we obtain immediately that in the low temperature $T = 1/\beta \ll 1$ limit,

$$\mathcal{F}_{\text{ren}} \sim -\gamma \omega_{\alpha,\Lambda} \frac{(-1)^{\frac{1}{2\alpha}}}{2\pi} m^{\frac{1}{\alpha}} \left(\psi\left(\frac{1}{2\alpha} + 1 \right) - \psi(1) \right) + \gamma(1 - \omega_{\alpha,\Lambda}) \frac{m^{\frac{1}{\alpha}}}{2 \sin \frac{\pi}{2\alpha}}$$

$$- \gamma \sum_{k=1}^{\infty} \frac{(-1)^k}{k} m^{-2k} (2\pi)^{2\alpha k} \zeta_R(-2\alpha k) \beta^{-2\alpha k-1}, \tag{5.21}$$

whereas in the high temperature $T = 1/\beta \gg 1$ limit,

$$\mathcal{F}_{\text{ren}} = \frac{1}{2\beta} \left\{ \gamma \log m^2 + 2\alpha\gamma \log \beta - 2\gamma \sum_{\substack{l \in \mathbb{N} \\ l \neq \frac{1}{2\alpha}}} \frac{(-1)^l}{l} m^{2l} \left(\frac{\beta}{2\pi} \right)^{2\alpha l} \zeta_R(2\alpha l) + \gamma \omega_{\alpha,\Lambda} (-1)^{\frac{1}{2\alpha}} \frac{\beta}{\pi} m^{\frac{1}{\alpha}} \right.$$

$$\left. \times \left[\alpha \left(\log \left(\frac{2\pi}{\beta m^{\frac{1}{\alpha}}} \right)^2 + 2\psi(1) \right) - \psi\left(\frac{1}{2\alpha} \right) + \psi(1) \right] \right\}. \tag{5.22}$$

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Figure 7. The renormalized free energy $\mathcal{F}_{\text{ren}}/m^{\frac{1}{\alpha}}$ as a function of $\beta m^{1/\alpha}$ when $\alpha = 0.1, 0.3, 0.5, 0.7, 0.9$ and $\gamma = 1$.

Note that (5.21) implies that if α is the reciprocal of an even integer, then when $T \to 0$, the leading order term of $\mathcal{F}_{\text{ren}}$ is

$$\mathcal{F}_{\text{ren}} \sim -\gamma \frac{(-1)^{\frac{1}{2\alpha}}}{2\pi} m^{\frac{1}{\alpha}} \left(\psi\left(\frac{1}{2\alpha} + 1 \right) - \psi(1) \right) + O(T^{2\alpha+1}), \qquad (5.23)$$

whose sign depends on α.

From (5.18) and (5.23), we see that in the low temperature limit, the leading order term of the renormalized Casimir free energy still depends linearly on γ, but its dependence on α is highly nontrivial.

Finally, we observe from (5.21) and (5.22) that the combination $\mathcal{F}_{\text{ren}}/m^{\frac{1}{\alpha}}$ depends on β and m in the combination $\beta m^{\frac{1}{\alpha}}$. The graphs and contour plots of $\mathcal{F}_{\text{ren}}/m^{\frac{1}{\alpha}}$ as a function of α and $\beta m^{\frac{1}{\alpha}}$ are shown in figures 7–9. The high temperature and low temperature behaviors of $\mathcal{F}_{\text{ren}}$ are shown in figure 10.

6. Concluding remarks

We have introduced a new Gaussian process called the fractional oscillator process with two indices, which is obtained as the solution to a stochastic differential equation with two fractional orders based on Riesz fractional derivative. Some basic properties of this process can be obtained based on the asymptotic properties of its covariance despite its complex nature. The main advantage of the fractional oscillator process parametrized by two indices over the fractional process with single index is that the former has its fractal dimension (or local self-similarity property) and the short-range dependence separately characterized by the two indices, while the latter has both these properties determined by a single index. Such a process may provide a more flexible model for applications in phenomena with short memory. Another advantage of the fractional oscillator process we introduce in this paper is the simple form of its spectral density, which makes it an ideal candidate for modeling physical processes. A careful discussion of the modeling issue would require the treatment of a whole paper. Therefore we defer it to a future publication.

Contour plot of the function $F_{ren}/m^{1/\alpha}$ as a function of $\beta\, m^{1/\alpha}$ and
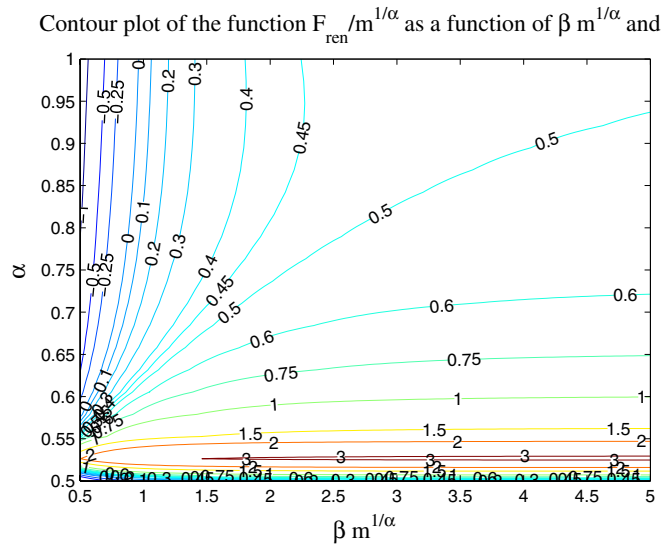
Figure 8. Contour plot of the renormalized free energy $\mathcal{F}_{ren}/m^{\frac{1}{\alpha}}$ as a function of $\beta m^{1/\alpha}$ and α. Here $\gamma = 1$.
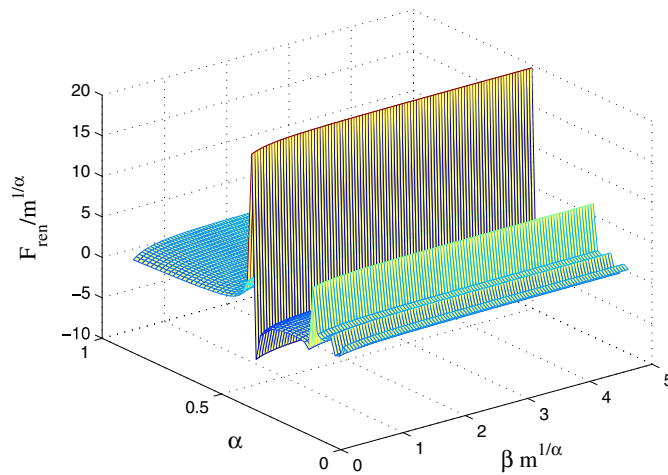
Figure 9. The renormalized free energy $\mathcal{F}_{ren}/m^{\frac{1}{\alpha}}$ as a function of $\beta m^{1/\alpha}$ and α. Here $\gamma = 1$.

We have also considered the possibility of regarding the fractional oscillator process as the velocity process of an anomalous diffusion. The formal extension of the fluctuation–dissipation relation and Einstein relation is discussed. By analogy regarding the fractional oscillator process as the Euclidean fractional scalar Klein–Gordon field in one dimension, we carry out the stochastic quantization of such a field with a nonlocal action. The Casimir energy associated with the fractional Klein–Gordon field at finite positive temperature was calculated by using the thermal zeta function regularization technique. The expression for the free energy has a rather complicated form. We thus consider the low and high temperature limits for the free energy. Graphical representations of this asymptotic behavior of the free energy are given.
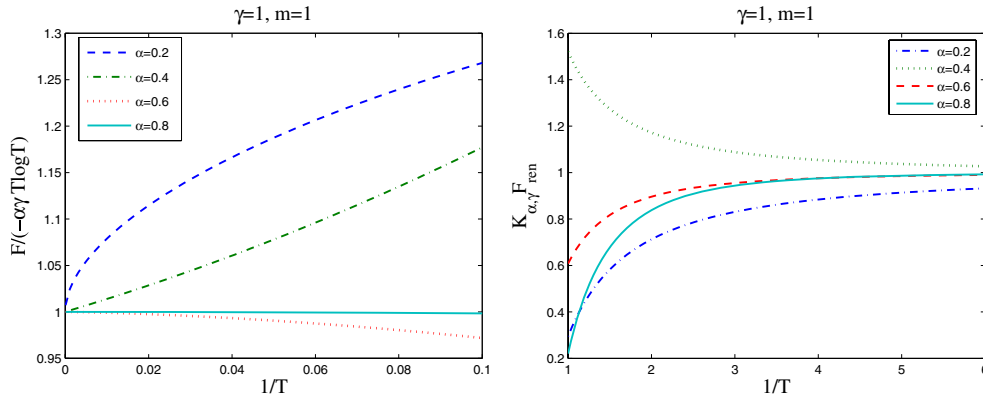
Figure 10. Left: this figure shows the ratio of the renormalized free energy $\mathcal{F}_{\text{ren}}$ to $-\alpha\gamma T \log T$ when $\beta = 1/T$ is small. Here $\gamma = 1$ and $m = 1$. Right: this figure shows the small T (large β) behavior of the renormalized free energy $\mathcal{F}_{\text{ren}}$. Here $K_{\alpha,\gamma} = 2\sin\frac{\pi}{2\alpha}/(\gamma m^{1/\alpha})$ and $\gamma = m = 1$.

Extension of our results can be carried out to give the n-dimensional Euclidean fractional Klein–Gordon field. However, the derivation of the asymptotic properties for the covariance function and the Casimir energy will be more complicated. The sign dependence of the free energy has important physical implication when the fractional quantum field under consideration is confined between parallel plates or cavities as the sign of the Casimir energy will determine whether the associated Casimir force is attractive or repulsive.

Recently, processes with variable local regularity and memory have attracted considerable attention. Multifractional processes such as multifractional Brownian motion [68, 69] have been defined which can be used to model multifractal processes. As a result, it would be natural to consider the fractional oscillator process of variable order, with α and γ being extended to time-dependent $\alpha(t)$ and $\gamma(t)$. We expect that the variable short-range dependence property remains valid, and the result for fractal dimension holds only locally. However, the Casimir energy calculation will require new mathematical techniques and approximations. Such a generalization may find applications for complex systems where the physical phenomena can have variable short memory and the fractal dimension varies with time or position.

Acknowledgments

Appendix A. The long-time asymptotic behavior of the mean-square displacement $\langle Y_{\alpha,\gamma}(t)^2 \rangle$

A.1. Ordinary derivative

We show that if the position process $Y_{\alpha,\gamma}(t)$ is related to the velocity process $X_{\alpha,\gamma}(t)$ by

$$X_{\alpha,\gamma}(t) = \frac{dY_{\alpha,\gamma}(t)}{dt},$$

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then the long-time behavior of the covariance function $C_{\alpha,\gamma}(\tau)$ of $X_{\alpha,\gamma}(t)$ does not show up in the leading term in the long-time asymptotic expression of the variance $\langle [Y_{\alpha,\gamma}(t)]^2 \rangle$. Its effect only appears in the second leading term. In fact, from (4.9), we have

$$\langle [Y_{\alpha,\gamma}(t)]^2 \rangle = 2 \int_0^t (t - \tau) C_{\alpha,\gamma}(\tau) \, d\tau = 2 \left[\int_0^t C_{\alpha,\gamma}(\tau) \, d\tau \right] t - 2 \int_0^t \tau C_{\alpha,\gamma}(\tau) \, d\tau.$$

Equation (3.18) gives us the long-time behavior of $C_{\alpha,\gamma}(\tau)$ as $C_{\alpha,\gamma}(\tau) \sim A_1 \tau^{-2\alpha-1} + O(\tau^{-4\alpha-1})$ for some constant A_1, which implies that the integrals $\int_0^\infty C_{\alpha,\gamma}(\tau) \, d\tau$ are convergent; and

$$\int_0^t C_{\alpha,\gamma}(\tau) \, d\tau = \int_0^\infty C_{\alpha,\gamma}(\tau) \, d\tau - \int_t^\infty C_{\alpha,\gamma}(\tau) \, d\tau \sim A_2 + A_3 t^{-2\alpha} + O(t^{-4\alpha}) \tag{A.1}$$

as $t \to \infty$. On the other hand, the integral $\int_0^\infty \tau C_{\alpha,\gamma}(\tau) \, d\tau$ is convergent if and only if $\alpha > 1/2$. In this case

$$\int_0^t \tau C_{\alpha,\gamma}(\tau) \, d\tau = \int_0^\infty \tau C_{\alpha,\gamma}(\tau) \, d\tau - \int_t^\infty \tau C_{\alpha,\gamma}(\tau) \, d\tau \sim A_4 + O(t^{-2\alpha+1}) \tag{A.2}$$

as $t \to \infty$. If $\alpha < 1/2$, then

$$\int_0^t \tau C(\tau) \, d\tau \sim A_5 t^{1-2\alpha} + O(t^{\max\{0, 1-4\alpha\}}) \tag{A.3}$$

as $t \to \infty$. In the borderline case $\alpha = 1/2$,

$$\int_0^t \tau C(\tau) \, d\tau \sim A_6 \log t + O(1). \tag{A.4}$$

Therefore, as $t \to \infty$, if $\alpha > 1/2$, then

$$\langle [Y_{\alpha,\gamma}(t)]^2 \rangle \sim [2A_2] t - [2A_4] + O(t^{1-2\alpha}).$$

If $\alpha = 1/2$, then

$$\langle [Y_{\alpha,\gamma}(t)]^2 \rangle \sim [2A_2] t - [2A_6] \log t + O(1).$$

Finally, if $\alpha < 1/2$, then

$$\langle [Y_{\alpha,\gamma}(t)]^2 \rangle \sim [2A_2] t + [2A_3 - 2A_5] t^{1-2\alpha} + O(t^{\max\{0, 1-4\alpha\}}).$$

These show that the leading term of $\langle [Y_{\alpha,\gamma}(t)]^2 \rangle$ is of order t, independent of α; and the second leading term is of order $t^{\max\{0, 1-2\alpha\}} \log t$, which depends on α.

A.2. Fractional derivative

We show that if the position process $Y_{\alpha,\gamma}(t)$ is related to the velocity process $X_{\alpha,\gamma}(t)$ by

$$X_{\alpha,\gamma}(t) = {}_0D_t^\chi Y_{\alpha,\gamma}(t), \qquad \tfrac{1}{2} < \chi < \tfrac{3}{2},$$

and ${}_0D_t^{\chi-j} Y_{\alpha,\gamma}(t) \big|_{t=0} = 0$ for $j = 1$ if $\chi \leqslant 1$ and $j = 1, 2$ if $\chi > 1$, then

$$\langle [Y_{\alpha,\gamma}(t)]^2 \rangle = 2B\lambda^{-2\alpha\gamma} \left[\frac{t^{2\chi-1}}{(2\chi-1)\Gamma(\chi)^2} \right] + O(t^{\max\{0, 2\chi-2, 2\chi-2\alpha-1\}} \log t). \tag{A.5}$$

From (4.12), we have

$$\langle [Y_{\alpha,\gamma}(t)]^2 \rangle = \frac{1}{\Gamma(\chi)^2} \int_0^t \int_0^t (t - s_1)^{\chi-1} (t - s_2)^{\chi-1} \langle X_{\alpha,\gamma}(s_1) X_{\alpha,\gamma}(s_2) \rangle \, ds_1 \, ds_2$$

$$= \frac{1}{\Gamma(\chi)^2} \int_0^t \int_0^t (t - s_1)^{\chi-1} (t - s_2)^{\chi-1} C_{\alpha,\gamma}(|s_1 - s_2|) \, ds_1 \, ds_2.$$

Using some calculus, this gives

$$
\langle [Y_{\alpha,\gamma}(t)]^2 \rangle = \frac{2}{\Gamma(\chi)^2} \int_0^t \int_0^{s_2} (t-s_1)^{\chi-1}(t-s_2)^{\chi-1} C_{\alpha,\gamma}(s_2-s_1)\, \mathrm{d}s_1\, \mathrm{d}s_2
$$

$$
= \frac{2}{\Gamma(\chi)^2} \int_0^t \int_0^s (t+\tau-s)^{\chi-1}(t-s)^{\chi-1} C_{\alpha,\gamma}(\tau)\, \mathrm{d}\tau\, \mathrm{d}s
$$

$$
= \frac{2}{\Gamma(\chi)^2} \int_0^t \left[\int_\tau^t (t+\tau-s)^{\chi-1}(t-s)^{\chi-1} \mathrm{d}s \right] C_{\alpha,\gamma}(\tau)\, \mathrm{d}\tau
$$

$$
= \frac{2}{\Gamma(\chi)^2} \int_0^t \left[\int_\tau^t u^{\chi-1}(u-\tau)^{\chi-1}\, \mathrm{d}u \right] C_{\alpha,\gamma}(\tau)\, \mathrm{d}\tau.
$$

The case $\chi = 1$ has been considered above. Now we consider the cases $\chi \in (1/2, 1)$ and $\chi \in (1, 3/2)$ separately. If $\chi \in (1/2, 1)$, writing

$$
(u-\tau)^{\chi-1} = u^{\chi-1} - (u^{\chi-1} - (u-\tau)^{\chi-1}),
$$

we have

$$
\int_\tau^t u^{\chi-1}(u-\tau)^{\chi-1}\, \mathrm{d}u = \frac{1}{2\chi-1}(t^{2\chi-1} - \tau^{2\chi-1}) - \int_\tau^t u^{\chi-1}(u^{\chi-1} - (u-\tau)^{\chi-1})\, \mathrm{d}u.
$$

Therefore, if $\chi \in (1/2, 1)$,

$$
\langle [Y_{\alpha,\gamma}(t)]^2 \rangle = \frac{2}{\Gamma(\chi)^2} \left(\frac{t^{2\chi-1}}{2\chi-1} \int_0^t C_{\alpha,\gamma}(\tau)\, \mathrm{d}\tau - \frac{1}{2\chi-1} \int_0^t \tau^{2\chi-1} C_{\alpha,\gamma}(\tau)\, \mathrm{d}\tau \right.
$$

$$
\left. - \int_0^t \left[\int_\tau^t u^{\chi-1}(u^{\chi-1} - (u-\tau)^{\chi-1})\, \mathrm{d}u \right] C_{\alpha,\gamma}(\tau)\, \mathrm{d}\tau \right).
$$

On the other hand, if $\chi \in (1, 3/2)$, writing

$$
(u-\tau)^{\chi-1} = u^{\chi-1} - (\chi-1)u^{\chi-2}\tau - [u^{\chi-1} - (\chi-1)u^{\chi-2}\tau - (u-\tau)^{\chi-1}],
$$

we have

$$
\int_\tau^t u^{\chi-1}(u-\tau)^{\chi-1}\, \mathrm{d}u = \frac{t^{2\chi-1}}{2\chi-1} - \frac{\tau}{2}t^{2\chi-2} - \frac{3-2\chi}{2(2\chi-1)}\tau^{2\chi-1}
$$

$$
- \int_\tau^t u^{\chi-1}\left[u^{\chi-1} - (\chi-1)u^{\chi-2}\tau - (u-\tau)^{\chi-1} \right] \mathrm{d}u.
$$

Therefore, if $\chi \in (1, 3/2)$

$$
\langle [Y_{\alpha,\gamma}(t)]^2 \rangle = \frac{2}{\Gamma(\chi)^2} \left(\frac{t^{2\chi-1}}{2\chi-1} \int_0^t C_{\alpha,\gamma}(\tau)\, \mathrm{d}\tau - \frac{t^{2\chi-2}}{2} \int_0^t \tau C_{\alpha,\gamma}(\tau)\, \mathrm{d}\tau \right.
$$

$$
- \frac{3-2\chi}{2(2\chi-1)} \int_0^t \tau^{2\chi-1} C_{\alpha,\gamma}(\tau)\, \mathrm{d}\tau
$$

$$
\left. - \int_0^t \left\{ \int_\tau^t u^{\chi-1}[u^{\chi-1} - (\chi-1)u^{\chi-2}\tau - (u-\tau)^{\chi-1}]\, \mathrm{d}u \right\} C_{\alpha,\gamma}(\tau)\, \mathrm{d}\tau \right).
$$

The large-t behaviors of $\int_0^t C_{\alpha,\gamma}(\tau)\, \mathrm{d}\tau$ and $\int_0^t \tau C_{\alpha,\gamma}(\tau)\, \mathrm{d}\tau$ have been studied and given by (A.1)–(A.4). For the term $\int_0^t \tau^{2\chi-1} C_{\alpha,\gamma}(\tau)\, \mathrm{d}\tau$, since $C_{\alpha,\gamma}(\tau)(\tau) \sim A_1\tau^{-2\alpha-1} + O(\tau^{-4\alpha-1})$ as $\tau \to \infty$, we find that if $2\chi - 1 < 2\alpha$,

$$
\int_0^t \tau^{2\chi-1} C_{\alpha,\gamma}(\tau)\, \mathrm{d}\tau = \int_0^\infty \tau^{2\chi-1} C_{\alpha,\gamma}(\tau)\, \mathrm{d}\tau - \int_t^\infty \tau^{2\chi-1} C_{\alpha,\gamma}(\tau)\, \mathrm{d}\tau
$$

$$
= B_\chi + O(t^{-2\alpha+2\chi-1}).
$$

However, if $2\chi - 1 > 2\alpha$,

$$\int_0^t \tau^{2\chi-1} C_{\alpha,\gamma}(\tau)\,d\tau = B_\chi t^{2\chi-2\alpha-1} + O(t^{\max\{0,2\chi-1-4\alpha\}}),$$

and if $2\chi - 1 = 2\alpha$, then

$$\int_0^t \tau^{2\chi-1} C_{\alpha,\gamma}(\tau)\,d\tau = B_\chi \log t + O(1).$$

Now for the term

$$\int_0^t \left[\int_\tau^t u^{\chi-1}(u^{\chi-1} - (u-\tau)^{\chi-1})\,du\right] C_{\alpha,\gamma}(\tau)\,d\tau,$$

By making a change of variable $u \mapsto u\tau$, we have

$$\int_0^t \left[\int_\tau^t u^{\chi-1}(u^{\chi-1} - (u-\tau)^{\chi-1})\,du\right] C_{\alpha,\gamma}(\tau)\,d\tau$$
$$= \int_0^t \tau^{2\chi-1}\left[\int_1^{\frac{t}{\tau}} u^{\chi-1}(u^{\chi-1} - (u-1)^{\chi-1})\,du\right] C_{\alpha,\gamma}(\tau)\,d\tau$$
$$= \int_1^\infty u^{\chi-1}(u^{\chi-1} - (u-1)^{\chi-1}) \int_0^{\frac{t}{u}} \tau^{2\chi-1} C_{\alpha,\gamma}(\tau)\,d\tau\,du.$$

Using the fact that $C_{\alpha,\gamma}(\tau) > 0$, we find that this term is bounded above by

$$\int_1^\infty u^{\chi-1}(u^{\chi-1} - (u-1)^{\chi-1})\,du \int_0^t \tau^{2\chi-1} C_{\alpha,\gamma}(\tau)(\tau)\,d\tau.$$

Note that $\chi \in (1/2, 1)$ implies that the first integral is convergent. Similarly, we have for $\chi \in (1, 3/2)$,

$$\int_0^t \left\{\int_\tau^t u^{\chi-1}[u^{\chi-1} - (\chi-1)u^{\chi-2}\tau - (u-\tau)^{\chi-1}]\,du\right\} C_{\alpha,\gamma}(\tau)\,d\tau$$
$$\leqslant \int_1^\infty u^{\chi-1}[u^{\chi-1} - (\chi-1)u^{\chi-2} - (u-1)^{\chi-1}]\,du \int_0^t \tau^{2\chi-1} C_{\alpha,\gamma}(\tau)\,d\tau,$$

and $\chi \in (1, 3/2)$ guarantees the convergence of the first integral. Gathering the results, we find that when $t \to \infty$,

$$\langle[Y_{\alpha,\gamma}(t)]^2\rangle \sim \mathcal{A}t^{2\chi-1} + O(\max\{t^{2\chi-2\alpha-1}, t^{2\chi-2}, t^{2\chi-2}\log t, t^0, t^0 \log t\}),$$

where

$$\mathcal{A} = \frac{2}{(2\chi-1)\Gamma(\chi)^2} \int_0^\infty C_{\alpha,\gamma}(\tau)\,d\tau = \frac{2B\lambda^{-2\alpha\gamma}}{(2\chi-1)\Gamma(\chi)^2},$$

and (A.5) follows.

Appendix B. The finite temperature free energy of the fractional Klein–Gordon field $\phi_{\alpha,\gamma}^T(t)$

We want to compute the free energy (5.13). We use standard techniques and write

$$2\sum_{n=1}^\infty \{[an]^{2\alpha} + m^2\}^{-\gamma s} = \frac{2}{\Gamma(\gamma s)} \int_0^\infty t^{\gamma s-1} \sum_{n=1}^\infty e^{-t([an]^{2\alpha}+m^2)}\,dt. \tag{B.1}$$

Since the obstacle to this integral to define an analytic function in s comes from the singularity at $t = 0$ of the integrand, the asymptotic behavior of

$$2 \sum_{n=1}^{\infty} e^{-t[an]^{2\alpha}}$$

as $t \to 0$ becomes crucial here. Using the representation

$$e^{-z} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(w) z^{-w} \, dw, \qquad c \in \mathbb{R}^+, \tag{B.2}$$

we have

$$2 \sum_{n=1}^{\infty} e^{-t[an]^{2\alpha}} = \frac{2}{2\pi i} \int_{c-i\infty}^{c+i\infty} dw \, \Gamma(w) t^{-w} a^{-2\alpha w} \zeta_R(2\alpha w),$$

when $c > 1/(2\alpha)$. Here $\zeta_R(s)$ is the Riemann zeta function. This gives the asymptotic behavior

$$2 \sum_{n=1}^{\infty} e^{-t[an]^{2\alpha}} \sim \frac{1}{\alpha} \Gamma\left(\frac{1}{2\alpha}\right) t^{-\frac{1}{2\alpha}} a^{-1} + 2 \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} t^k a^{2\alpha k} \zeta_R(-2\alpha k) \tag{B.3}$$

as $t \to 0$ (also as $a \to 0$). Using the fact that $\zeta_R(0) = -1/2$, then with

$$K(t) = 2 \sum_{n=1}^{\infty} e^{-t[an]^{2\alpha}} - \frac{1}{\alpha} \Gamma\left(\frac{1}{2\alpha}\right) t^{-\frac{1}{2\alpha}} a^{-1} + 1,$$

equation (B.3) implies that $K(t) = O(t)$ as $t \to 0$. Now, we can continue the evaluation of the integral in (B.1):

$$\frac{2}{\Gamma(\gamma s)} \int_0^{\infty} t^{\gamma s - 1} \sum_{n=1}^{\infty} e^{-t([an]^{2\alpha} + m^2)} \, dt = \frac{1}{\Gamma(\gamma s)} \int_0^{\infty} t^{\gamma s - 1} \left[\frac{1}{\alpha} \Gamma\left(\frac{1}{2\alpha}\right) t^{-\frac{1}{2\alpha}} a^{-1} - 1 \right] e^{-tm^2} \, dt$$

$$+ \frac{1}{\Gamma(\gamma s)} \int_0^{\infty} t^{\gamma s - 1} K(t) e^{-tm^2} \, dt$$

$$= \frac{1}{\alpha} \frac{\Gamma\left(\frac{1}{2\alpha}\right) \Gamma\left(\gamma s - \frac{1}{2\alpha}\right)}{\Gamma(\gamma s)} a^{-1} m^{-2\gamma s + \frac{1}{\alpha}} - m^{-2\gamma s} + \frac{1}{\Gamma(\gamma s)} \int_0^{\infty} t^{\gamma s - 1} K(t) e^{-tm^2} \, dt.$$

Since $K(t) = O(t)$ as $t \to 0$, the second integral in the last line of the above equation defines an analytic function for $s > -1/\gamma$. Combining with the first term in (5.14), we find that an analytic continuation of $\zeta(s)$ to $\mathrm{Re}\, s > -1/\gamma$ is given by

$$\zeta(s) = \frac{\Gamma\left(\frac{1}{2\alpha}\right) \Gamma\left(\gamma s - \frac{1}{2\alpha}\right)}{\alpha \Gamma(\gamma s)} a^{-1} m^{-2\gamma s + \frac{1}{\alpha}} + \frac{1}{\Gamma(\gamma s)} \int_0^{\infty} t^{\gamma s - 1} K(t) e^{-tm^2} \, dt. \tag{B.4}$$

To evaluate $\zeta(0)$ and $\zeta'(0)$, we observe that

$$\int_0^{\infty} t^{\gamma s - 1} K(t) e^{-tm^2} \, dt$$

is analytic for $s > -1/\gamma$. Therefore the only possible contribution to $\zeta(0)$ comes from the first term in (B.4) when $1/(2\alpha) \in \mathbb{N}$. Denote by Λ the set

$$\Lambda = \left\{ \frac{1}{2u} : u \in \mathbb{N} \right\},$$

and let

$$\omega_{\alpha,\Lambda} = \begin{cases} 1, & \text{if} \quad \alpha \in \Lambda \\ 0, & \text{if} \quad \alpha \notin \Lambda. \end{cases}$$

Then we find that

$$\zeta(0) = 2\omega_{\alpha,\Lambda}(-1)^{\frac{1}{2\alpha}}a^{-1}m^{\frac{1}{\alpha}},$$

and

$$\zeta'(0) = 2\omega_{\alpha,\Lambda}(-1)^{\frac{1}{2\alpha}}a^{-1}m^{\frac{1}{\alpha}}\gamma\left\{\psi\left(\frac{1}{2\alpha}+1\right)-\psi(1)-\log m^2\right\}$$

$$-\gamma(1-\omega_{\alpha,\Lambda})\frac{2\pi}{\sin\frac{\pi}{2\alpha}}a^{-1}m^{\frac{1}{\alpha}}+\gamma\int_0^\infty t^{-1}K(t)\,\mathrm{e}^{-tm^2}\,\mathrm{d}t.$$

Here $\psi(z) = \Gamma'(z)/\Gamma(z)$ is the logarithmic derivative of the gamma function. Substituting the above into (5.13) gives the Casimir free energy

$$\mathcal{F} = \frac{1}{2\beta}\left\{\omega_{\alpha,\Lambda}(-1)^{\frac{1}{2\alpha}}\frac{\beta}{\pi}m^{\frac{1}{\alpha}}\left[\log\mu^2-\gamma\left(\psi\left(\frac{1}{2\alpha}+1\right)-\psi(1)-\log m^2\right)\right]\right.$$

$$\left.+\gamma(1-\omega_{\alpha,\Lambda})\frac{\beta}{\sin\frac{\pi}{2\alpha}}m^{\frac{1}{\alpha}}-\gamma\int_0^\infty t^{-1}K(t)\,\mathrm{e}^{-tm^2}\,\mathrm{d}t\right\}.$$

To study the low temperature asymptotic behavior of $\mathcal{F}$, we first use (B.3) to obtain asymptotic behavior of $\zeta(s)$ when $a \to 0$:

$$\zeta(s) \sim \frac{\Gamma\left(\frac{1}{2\alpha}\right)\Gamma\left(\gamma s-\frac{1}{2\alpha}\right)}{\alpha\Gamma(\gamma s)}a^{-1}m^{-2\gamma s+\frac{1}{\alpha}}+2\sum_{k=1}^\infty\frac{(-1)^k}{k!}\frac{\Gamma(\gamma s+k)}{\Gamma(\gamma s)}m^{-2\gamma s-2k}a^{2\alpha k}\zeta_R(-2\alpha k).$$

From this we can find the asymptotic behavior of $\zeta(0)$ and $\zeta'(0)$, which, when substituted into (5.13) gives

$$\mathcal{F} \sim \omega_{\alpha,\Lambda}\frac{(-1)^{\frac{1}{2\alpha}}}{2\pi}m^{\frac{1}{\alpha}}\left[\log\mu^2-\gamma\left(\psi\left(\frac{1}{2\alpha}+1\right)-\psi(1)-\log m^2\right)\right]$$

$$+\gamma(1-\omega_{\alpha,\Lambda})\frac{m^{\frac{1}{\alpha}}}{2\sin\frac{\pi}{2\alpha}}-\gamma\sum_{k=1}^\infty\frac{(-1)^k}{k}m^{-2k}(2\pi)^{2\alpha k}\zeta_R(-2\alpha k)\beta^{-2\alpha k-1} \quad \text{(B.5)}$$

when $\beta \to \infty$. When $\alpha = 1$, we can find an exact formula as follows. By applying the Jacobi inversion formula

$$1+2\sum_{n=1}^\infty\mathrm{e}^{-t[an]^2} = \frac{1}{a}\sqrt{\frac{\pi}{t}}\sum_{n=-\infty}^\infty\mathrm{e}^{-\frac{\pi^2}{ta^2}n^2},$$

we have

$$\zeta(s) = \frac{\sqrt{\pi}}{a\Gamma(\gamma s)}\int_0^\infty t^{\gamma s-\frac{1}{2}-1}\sum_{n=-\infty}^\infty\mathrm{e}^{-\frac{\pi^2}{ta^2}n^2-tm^2}\,\mathrm{d}t$$

$$= \frac{\sqrt{\pi}\Gamma\left(\gamma s-\frac{1}{2}\right)}{a\Gamma(\gamma s)}m^{1-2\gamma s}+\frac{4\sqrt{\pi}}{a\Gamma(\gamma s)}\sum_{n=1}^\infty\left(\frac{\pi n}{am}\right)^{\gamma s-\frac{1}{2}}K_{\gamma s-\frac{1}{2}}\left(\frac{2\pi nm}{a}\right).$$

Here $K_\nu(z)$ is the modified Bessel function of second kind. Together with $K_{1/2}(z) = \sqrt{\pi/(2z)}\,\mathrm{e}^{-z}$, one obtains

$$\mathcal{F} = \frac{\gamma m}{2}-\frac{\gamma}{\beta}\sum_{n=1}^\infty\frac{1}{n}\mathrm{e}^{-\beta nm} = \frac{\gamma m}{2}+\frac{\gamma}{\beta}\log(1-\mathrm{e}^{-\beta m}) = \frac{\gamma}{\beta}\log\left[2\sinh\frac{\beta m}{2}\right]. \quad \text{(B.6)}$$

To study the high temperature behavior of $\mathcal{F}$, we can use the expansion

$$\exp(-tm^2) = \sum_{l=0}^\infty\frac{(-1)^l}{l!}t^l m^{2l} \quad \text{(B.7)}$$

and find that

$$2 \sum_{n=1}^{\infty} \{[an]^{2\alpha} + m^2\}^{-\gamma s} = \frac{2}{\Gamma(\gamma s)} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} m^{2l} \int_0^{\infty} t^{\gamma s + l - 1} \sum_{n=1}^{\infty} e^{-t[an]^{2\alpha}} \, dt$$

$$= \frac{2}{\Gamma(\gamma s)} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} m^{2l} a^{-2\alpha(\gamma s + l)} \Gamma(\gamma s + l) \zeta_R(2\alpha(\gamma s + l)). \qquad (B.8)$$

The $l = 0$ term in (B.8) will contribute -1 to $\zeta(0)$, which is canceled by the contribution from the term $m^{-2\gamma s}$ (5.14). Moreover, since $\zeta_R(s)$ is meromorphic on $\mathbb{C}$ with a simple pole at $s = 1$ with

$$\zeta_R(s) = \frac{1}{s-1} - \psi(1) + O(s-1), \qquad \text{as} \quad s \to 1,$$

we see that if $\alpha = 1/(2j)$ for some $j \in \mathbb{N}$, then there is another nonzero contribution to $\zeta(0)$ arising from the $l = j$ term in (B.8). Therefore,

$$\zeta(0) = 2\omega_{\alpha,\Lambda}(-1)^{\frac{1}{2\alpha}} a^{-1} m^{\frac{1}{\alpha}}$$

and

$$\zeta'(0) = -2\gamma \log m - 2\alpha\gamma \log \frac{2\pi}{a} + 2\gamma \sum_{\substack{l \in \mathbb{N} \\ l \neq \frac{1}{2\alpha}}} \frac{(-1)^l}{l} m^{2l} a^{-2\alpha l} \zeta_R(2\alpha l)$$

$$- 2\omega_{\alpha,\Lambda}(-1)^{\frac{1}{2\alpha}} a^{-1} m^{\frac{1}{\alpha}} \gamma \left(\alpha[\log a^2 + 2\psi(1)] - \psi\left(\frac{1}{2\alpha}\right) + \psi(1) \right).$$

This gives us

$$\mathcal{F} = \frac{1}{2\beta} \left\{ \gamma \log m^2 + 2\alpha\gamma \log \beta - 2\gamma \sum_{\substack{l \in \mathbb{N} \\ l \neq \frac{1}{2\alpha}}} \frac{(-1)^l}{l} m^{2l} \left(\frac{\beta}{2\pi} \right)^{2\alpha l} \zeta_R(2\alpha l) + \omega_{\alpha,\Lambda}(-1)^{\frac{1}{2\alpha}} \frac{\beta}{\pi} m^{\frac{1}{\alpha}} \right.$$

$$\left. \times \left[\log \mu^2 + \alpha\gamma \left(\log\left(\frac{2\pi}{\beta} \right)^2 + 2\psi(1) \right) - \gamma \left[\psi\left(\frac{1}{2\alpha}\right) - \psi(1) \right] \right] \right\}. \qquad (B.9)$$

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